HANDOUT M. 1 - COMPLEX NUMBERS

NOTE: All the problems in this handout must be solved by hand using the analytical procedure. MATLAB must be used only to verify the result obtained.

What are complex numbers?

The set of complex numbers is represented in the standard form as

$\{a+bi\}$ where 'a' and 'b' are real numbers. The variable 'i' is defined as $i^2 = -1$.

We often use the variable z = a + bi to represent a complex number. The number 'a' is called the **real part** of z (Re z) while 'b' is called the **imaginary part** of z (Im z).

Properties of complex numbers

1. Complex numbers are also represented as ordered pairs. For example the complex number z = a + bi can be represented as an ordered pair (a, b) in the complex plane. This is a plane wherein the x-axis represents the real part and the y-axis represents the imaginary part. This is further explained with the help of the following figure.

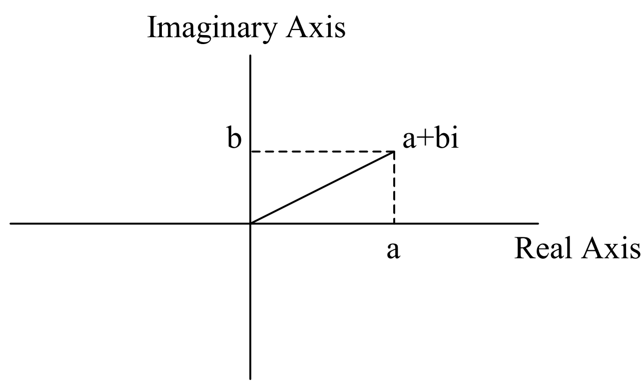

Figure 1. Representation of a complex number on a complex plane

2. Two complex numbers (a, b) and (c, d) are *equal* if and only if their real parts are equal and also their imaginary parts are equal, i.e. a = c and b = d.

So

$$a + bi = c + di$$
$$\Leftrightarrow$$
$$a = c \quad ; \quad b = d.$$

3. Basic operations with complex numbers:

a. Sum of complex numbers

$$(a+bi)+(c+di)=(a+c)+(b+d)i$$
$$(a+bi)-(c+di)=(a-c)+(b-d)i.$$

Example

$$(2+3i)+(3+4i)=(5+7i)$$
$$(2+3i)-(3+4i)=(-1-1i).$$

b. Product of complex numbers

$$(a+bi)\times(c+di)=(ac+adi+bci+bdi^2)$$
$$=(ac-bd)+(ad+bc)i.$$

Note that the relation $i^2=-1$ is used to get the final result.

Example

$$(2+4i)\times(4+5i)=(8+10i+16i+20i^2)$$
$$=(-12+26i).$$

c. Magnitude of a complex number

The magnitude of a complex number z = a + bi is defined as

$$|z|=\sqrt{a^2+b^2}$$

d. Complex conjugates

The conjugate of a complex number z = a + bi is defined as

$$\bar{z}=\overline{a+bi}=a-bi$$

Note that the magnitude of the complex conjugate is the same as that of the complex number, ‘z’.

e. Properties of the complex conjugate.

$$z\bar{z}=(a+bi)\times(a-bi)=(a^2+b^2)=|z|^2$$

$$\text{Re}(z) = \frac{1}{2}(z + \bar{z})$$

$$\text{Im}(z) = \frac{1}{2i}(z - \bar{z})$$

If $z = c + di$, then the complex conjugate of z when viewed in the complex plane is given as

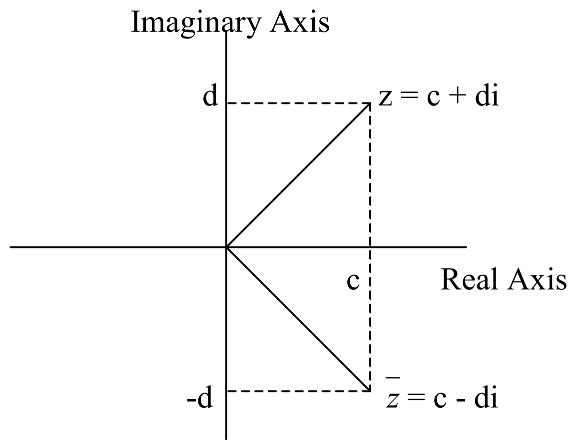

Figure 2. Representation of a complex conjugate

Example

- $z = (2 + 4i)$
  $then \quad \bar{z} = (2 - 4i).$

- Calculate the magnitude of the following complex numbers:

  $z_1 = 3 + 4i.$

  $z_2 = 5 - 6i.$

  $$|z_1| = \sqrt{(3)^2 + (4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

  $$|z_2| = \sqrt{(5)^2 + (-6)^2} = \sqrt{25 + 36} = \sqrt{61}$$

f. Division of complex numbers

Let us suppose that, we need to divide two complex numbers $z_1 = a + bi$ and $z_2 = c + di$. In dividing the complex numbers, we rationalize the denominator using the fact that

$$(c+di)(c-di)=(c^2+d^2)$$

So,

$$\frac{a+bi}{c+di}=\frac{a+bi}{c+di}\times\frac{c-di}{c-di}=\frac{(ac+bd)}{(c^2+d^2)}+\frac{(ad-bc)}{(c^2+d^2)}i.$$

Example

Divide the complex numbers z_1 = 2 + 3i and z_2 = 3 + 4i

$$\frac{z_1}{z_2}=\frac{2+3i}{3+4i}=\frac{2+3i}{3+4i}\times\frac{3-4i}{3-4i}=\frac{(6-8i+9i-12i^2)}{9+16}=\frac{18}{25}+\frac{1}{25}i.$$

Polar form of complex numbers

For z = a + bi

Let

$$\begin{aligned} a &= r\cos\theta \\ b &= r\sin\theta. \end{aligned} \tag{1}$$

Squaring equation (1a) and equation (1b) and then adding, we get

$$r=\sqrt{a^2+b^2}=|z|.$$

Dividing equation (1a) by equation (1b) gives

$$\tan\theta=\frac{b}{a}$$

'r' is called the magnitude of the complex number and 'θ' is called the argument of the complex number 'z'. The argument of the complex number is represented as

θ = arg(z).

So 'z' can be rewritten as

$$z=r\cos\theta+r\sin\theta\ i.$$

By Euler's equation, the polar form can be represented as

$$z = (r\cos\theta) + (r\sin\theta)i = re^{i\theta}$$

[Note that $e^{i\theta} = \cos\theta \; + i\sin\theta$].

The following figure shows the representation of a complex number, 'z' in its polar form in the complex plane.

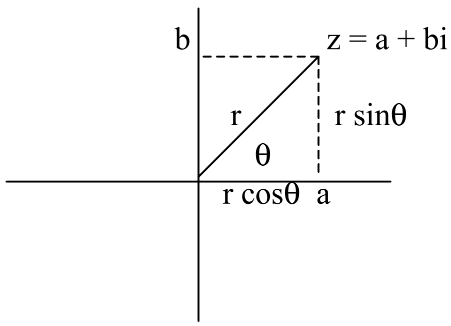

Figure 3. Polar form of a complex number

Examples

- $e^{i\pi} = (\cos\pi) + (\sin\pi)i = (-1) + (0)i = -1$
- $2e^{i\frac{\pi}{2}} = (2\cos\frac{\pi}{2}) + (2\sin\frac{\pi}{2})i = (0) + (2)i = 2i$
- Given $z = 1 + i$, represent the complex number in the polar form.

  Since z = 1 + i, a = 1 and b = 1. Hence

  $$r = \sqrt{a^2 + b^2} = \sqrt{1^1 + 1^1} = \sqrt{2}$$
  $$\tan\theta = \frac{b}{a} = \frac{1}{1} = 1$$
  $$\Rightarrow \theta = \tan^{-1}(1) = \frac{\pi}{4}$$

  Therefore the complex number in the polar form is given by

  $$z = 1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$$

Multiplication and division of complex numbers in polar form

If

$$z_1 = r_1 e^{i\theta_1} \qquad and \qquad z_2 = r_2 e^{i\theta_2}$$

then

$$z_1 z_2 = r_1 r_2 e^{i\theta_1} e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}.$$

Note: The conjugate of a complex number $z = re^{i\theta}$ in the polar form is represented by

$$\bar{z} = re^{-i\theta}$$

Note: Doing the above calculations with complex numbers in Cartesian form is much more cumbersome.

Using MATLAB to perform various operations on complex numbers

1. Given a complex number, to get the real and the imaginary part of the complex number use the '*real*' and the '*imag*' command.

To know more about the commands type the following command in the MATLAB command window.

```
help real

 REAL   Complex real part.
    REAL(X) is the real part of X.
help imag

 IMAG   Complex imaginary part.
    IMAG(X) is the imaginary part of X.
    See I or J to enter complex numbers.
```

Example

Determine the real and imaginary part of the following complex number using MATLAB

Z = 2 + 3i;

```
z=2+3i;
r = real(z)
i = imag(z)
```

Running the above code gives the following result

```
r =

     2


i =

     3
```

2. Given a complex number, to determine the magnitude and the argument of the complex number, use the '*abs*' and the '*angle*' command. Type the help command in the MATLAB command window to know more about the commands.

```
help abs

 ABS    Absolute value.
    ABS(X) is the absolute value of the elements of X. When
    X is complex, ABS(X) is the complex modulus (magnitude) of
    the elements of X.


help angle

 ANGLE  Phase angle.
    ANGLE(H) returns the phase angles, in radians, of a matrix with
    complex elements.
```

Example

Determine the magnitude and the argument of the complex number z = 1 + 2i.

Analytical method

$$\text{magnitude} = \sqrt{a^2 + b^2} = \sqrt{1^2 + 2^2} = \sqrt{1+4} = \sqrt{5} = 2.2361$$

$$\text{theta} = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{2}{1}\right) = \tan^{-1}(2) = 70.4833^\circ.$$

The MATLAB code for calculating the magnitude and the angle is as follows.

```
z=1+2i;
mag=abs(z)
theta=angle(z)
```

Running the above code, gives the following result

```
mag =

   2.2361
```

```
theta =

   1.1071
```

Note that, the value of theta is in **radians**.

3. To determine the conjugate of a complex number, the ‘*conj*’ command in MATLAB is used.

```
help conj

 CONJ   Complex conjugate.
    CONJ(X) is the complex conjugate of X.
    For a complex X, CONJ(X) = REAL(X) - i*IMAG(X).
```

Example

Determine the complex conjugate of the complex number z = 1 + 2i.

Analytical method

The complex conjugate of the above complex number is given by just changing the sign of the imaginary part of the complex number, i.e., $\bar{z} = 1 - 2i$.

The MATLAB code to verify the result is

```
z=1+2i;
c = conj(z)
```

The result of the above code is as follows

```
c =

   1.0    - 2.0000i
```

4. To convert Cartesian to polar and vice versa.

The ‘*cart2pol*’ and ‘*pol2cart*’ commands are used for the above-mentioned purpose. To know more about the commands, type the following commands in the MATLAB command window.

```
help cart2pol

 CART2POL Transform Cartesian to polar coordinates.
    [TH,R] = CART2POL(X,Y) transforms corresponding elements of data
    stored in Cartesian coordinates X,Y to polar coordinates (angle TH
    and radius R).  The arrays X and Y must be the same size (or
    either can be scalar). TH is returned in radians.
```

```
    [TH,R,Z] = CART2POL(X,Y,Z) transforms corresponding elements of
    data stored in Cartesian coordinates X,Y,Z to cylindrical
    coordinates (angle TH, radius R, and height Z).  The arrays X,Y,
    and Z must be the same size (or any of them can be scalar).  TH is
    returned in radians.
```

```
help pol2cart

 POL2CART Transform polar to Cartesian coordinates.
    [X,Y] = POL2CART(TH,R) transforms corresponding elements of data
    stored in polar coordinates (angle TH, radius R) to Cartesian
    coordinates X,Y.  The arrays TH and R must the same size (or
    either can be scalar).  TH must be in radians.

    [X,Y,Z] = POL2CART(TH,R,Z) transforms corresponding elements of
    data stored in cylindrical coordinates (angle TH, radius R, height
    Z) to Cartesian coordinates X,Y,Z. The arrays TH, R, and Z must be
    the same size (or any of them can be scalar).TH must be in radians.
```

This process is explained in detail with the help of an example.

Example

a. Convert the complex number z = 1 + 2i to polar form. The MATLAB code is as shown below.

Analytical method

$$|z| = \sqrt{(1)^2 + (2)^2} = \sqrt{5} = 2.2361$$

$$\theta = \tan^{-1}(\frac{2}{1}) = 1.1071 \ radians = 70.4833^{\circ}.$$

The MATLAB code to verify the result is

```
z=1+2i;
r = real(z);
i = imag(z);
[th,r] = cart2pol(r,i)
```

Note that the arguments of the '*cart2pol*' function must be the real and the imaginary parts of the complex number. The result of the above code is given below.

```
th =

    1.1071

r =

    2.2361
```

Note that theta is given in **radians**. Therefore the complex number in the polar form is

$$z = 1 + 2i = 2.2361e^{i(70.4833^{\circ})}.$$

b. Convert the complex number whose magnitude is 2 units and the argument is 90 degrees to Cartesian form.

Note that the argument must be always in **radians**. So 90 degrees in radians is π/2.

Analytical method

$$|z| = 2;$$

$$\theta = \frac{\pi}{2};$$

$$z = re^{i\theta} = 2e^{i\frac{\pi}{2}} = 2(\cos(\frac{\pi}{2}) + \sin(\frac{\pi}{2})i) = 2(0 + i) = 2i$$

The MATLAB code for this purpose is given below.

```
r=2;
th=pi/2;
[x,y]=pol2cart(th,r)
```

Running the above code gives the following result.

```
x =

  1.2246e-016


y =

     2
```

From the above result it can be seen that x is effectively zero. So the complex number in the Cartesian form is z = 2i.

KEY CONCEPTS

Cartesian form	*Transformations*	*Polar form*
$z = a + bi$	$a = r\cos\theta$	$z = re^{i\theta}$
$a = \mathrm{Re}(z)$	$b = r\sin\theta$	$r = \|z\|$
$b = \mathrm{Im}(z)$		$\theta = \arg(z)$
$\|z\| = \sqrt{a^2 + b^2}$	$r = \sqrt{a^2 + b^2}$	$\bar{z} = re^{-i\theta}$
$\bar{z} = a - bi$	$\tan\theta = \dfrac{b}{a}$	

Euler's equation

$$e^{i\theta} = \cos\theta + \sin\theta\, i$$

provides the connection between these two representations of complex numbers.

Assignment

1. Determine the magnitude and argument of the following complex numbers. Verify the result using MATLAB

a. 1 + 2i
b. 3 + 4i
c. 5 – 7i
d. 4 – 2i

2. Convert the following complex numbers to polar form. Verify the result using MATLAB.

a. 2 + 4i
b. 1 + 0i
c. 0 + 1i
d. 1 + 2i

3. Convert the following complex numbers to Cartesian form. Verify the result using MATLAB

a. r = 1; θ = 90 degrees
b. r = 2; θ = 0 radians
c. r = 2; θ = 30 degrees.

Recommended Reading

"Feedback Control of Dynamic Systems" Third Edition, by Gene F. Franklin et.al – pp 736 – 741.