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COMPLEX NUMBER ALGEBRA RELATING TO SINUSOIDAL FUNCTIONS

1) If $u(t) = A\cos(\omega\, t) + B\sin(\omega\, t)$, where A and B are real, then we can rewrite u(t) as:

$$u(t) = \frac{A}{2}e^{j\omega\, t} + \frac{A}{2}e^{-j\omega\, t} + \frac{B}{2j}e^{j\omega\, t} - \frac{B}{2j}e^{-j\omega\, t},$$
$$\Rightarrow u(t) = \frac{(A - Bj)}{2}e^{j\omega\, t} + \frac{(A + Bj)}{2}e^{-j\omega\, t}.$$

2) If $u(t) = M\cos(\omega\, t + \phi)$, where M and ϕ are real, then we can rewrite this as:

$$u(t) = \frac{M}{2}\left(\cos(\phi) + j\sin(\phi)\right)e^{j\omega\, t} + \frac{M}{2}\left(\cos(\phi) - j\sin(\phi)\right)e^{-j\omega\, t},$$
$$\Rightarrow u(t) = \left(\frac{M}{2}e^{j\phi}\right)e^{j\omega\, t} + \left(\frac{M}{2}e^{-j\phi}\right)e^{-j\omega\, t}.$$

Example

Solve the following differential equation for zero initial conditions.

$\dot{y}(t) + ay(t) = u(t)$, where $u(t) = A\cos(\omega\, t + \phi)$.

Sol:

Taking the Laplace transforms of the differential equation, we get

$$sY(s) + aY(s) = U(s),$$
$$\Rightarrow Y(s) = \frac{1}{(s + a)}U(s). \qquad (1)$$

$$u(t) = A\cos(\omega\, t + \phi),$$
$$\Rightarrow u(t) = \frac{A}{2}e^{j\phi}e^{j\omega\, t} + \frac{A}{2}e^{-j\phi}e^{-j\omega\, t},$$
$$\Rightarrow U(s) = \left(\frac{A}{2}e^{j\phi}\right)\frac{1}{(s - j\omega)} + \left(\frac{A}{2}e^{-j\phi}\right)\frac{1}{(s + j\omega)}.$$

Substituting the value of U(s) in equation (1), we get

$$Y(s)=\left(\frac{A}{2}e^{j\phi}\right)\frac{1}{(s-j\omega)}\frac{1}{(s+a)}+\left(\frac{A}{2}e^{-j\phi}\right)\frac{1}{(s+j\omega)}\frac{1}{(s+a)}. \qquad (2)$$

To solve for y(t), we'll use the partial fraction expansion of $\frac{1}{(s-j\omega)}\frac{1}{(s+a)}$ and $\frac{1}{(s+j\omega)}\frac{1}{(s+a)}$.

$$\frac{1}{(s-j\omega)}\frac{1}{(s+a)}=\frac{\left.\frac{1}{(s+a)}\right|_{s=j\omega}}{(s-j\omega)}+\frac{\left.\frac{1}{(s-j\omega)}\right|_{s=-a}}{(s+a)}=\frac{1}{(j\omega\ +a)}\frac{1}{(s-j\omega)}+\frac{1}{(-a-j\omega)}\frac{1}{(s+a)},$$

$$=G(j\omega)\frac{1}{(s-j\omega)}-G(j\omega)\frac{1}{(s+a)}.$$

Similarly,

$$\frac{1}{(s+j\omega)}\frac{1}{(s+a)}=\frac{\left.\frac{1}{(s+a)}\right|_{s=-j\omega}}{(s+j\omega)}+\frac{\left.\frac{1}{(s+j\omega)}\right|_{s=-a}}{(s+a)}=\frac{1}{(-j\omega\ +a)}\frac{1}{(s+j\omega)}+\frac{1}{(-a+j\omega)}\frac{1}{(s+a)},$$

$$=G(-j\omega)\frac{1}{(s+j\omega)}-G(-j\omega)\frac{1}{(s+a)}.$$

Substituting the partial fraction expansions in equation (2), we get

$$Y(s)=\left(\frac{A}{2}e^{j\phi}\right)G(j\omega)\frac{1}{(s-j\omega)}+\left(\frac{A}{2}e^{-j\phi}\right)G(-j\omega)\frac{1}{(s+j\omega)}$$

$$+\left[-\left(\frac{A}{2}e^{j\phi}\right)G(j\omega)-\left(\frac{A}{2}e^{-j\phi}\right)G(-j\omega)\right]\frac{1}{(s+a)},$$

$$\Rightarrow Y(s)=\left(\frac{A}{2}e^{j\phi}\right)G(j\omega)\frac{1}{(s-j\omega)}+\left(\frac{A}{2}e^{-j\phi}\right)G(-j\omega)\frac{1}{(s+j\omega)}+Q\frac{1}{(s+a)}. \qquad (3)$$

Taking the inverse Laplace transforms of equation (3), we get

$$y(t) = \frac{A}{2} e^{j\phi} G(j\omega) e^{j\omega t} + \frac{A}{2} e^{-j\phi} G(-j\omega) e^{-j\omega t} + Qe^{-at}, \tag{4}$$

where Q is always a Real number.

If we write,

$$G(j\omega) = G_\omega e^{j\theta_\omega},$$
$$G(-j\omega) = G_\omega e^{-j\theta_\omega},$$

then, we can write equation (4) in the form:

$$y(t) = AG_\omega \cos(\omega t + \phi + \theta_\omega) + Qe^{-at}. \tag{5}$$

For all $a > 0$, the term Qe^{-at} vanishes eventually. It can be concluded from equation (5) that y(t) has the same frequency as u(t), but the amplitude is multiplied by G_ω and the phase is shifted by θ_ω.