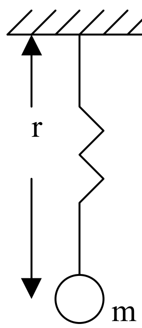
HANDOUT E.12 - EXAMPLES ON LINEARIZATION

Example 1

Consider the system shown below.



The governing differential equations of motion for the above system is given by

$$m\ddot{r} + kr - kl_0 - mr\dot{\theta}^2 - mg\cos\theta = 0 \tag{1}$$

$$mr\ddot{\theta} + 2m\dot{r}\dot{\theta} + mg\sin\theta = 0. \tag{2}$$

where, l_0 is the initial length of the spring and 'k' is the stiffness constant of the spring.

Note that the above differential equations are non-linear in nature. First, to find the equilibrium point, equate all the derivative terms to zero. Therefore equation (2) reduces to

$$mg\sin\theta = 0,$$

$$\Rightarrow \sin\theta = 0,$$

$$\Rightarrow \theta = n\pi.$$

There $\theta_0 = 0$ is one equilibrium point for the above system.

Following the same procedure for equation (1), we get

$$kr - kl_0 - mg\cos\theta = 0,$$

$$\Rightarrow kr - kl_0 - mg = 0,$$

$$\Rightarrow r = \frac{mg + kl_0}{k} = r_0. \tag{3}$$

Therefore $r = r_0$ is the equilibrium value for the variable 'r'.

Expanding each term in equation (1) by Taylor's series about the equilibrium point and neglecting the higher order terms, we have

$$m\ddot{r} + kr - kl_0 - mr\dot{\theta}^2 - mg\cos\theta = 0,$$

$$\Rightarrow m\ddot{r} + kr - kl_0 - \left(mr\dot{\theta}^2\right)\Big|_{\substack{r=r_0\\ \theta=\theta_0}} - \frac{\partial}{\partial r}(mr\dot{\theta}^2)\Big|_{\substack{r=r_0\\ \dot{\theta}=0}} \cdot (r - r_0) - \frac{\partial}{\partial \dot{\theta}}(mr\dot{\theta}^2)\Big|_{\substack{r=r_0\\ \dot{\theta}=0}} \cdot (\dot{\theta} - 0)$$

$$-(mg\cos\theta)\Big|_{\substack{r=r_0\\ \theta=\theta_0}} - \frac{\partial}{\partial\theta}(mg\cos\theta)\Big|_{\substack{r=r_0\\ \theta=0}} \cdot (\theta - \theta_0) = 0,$$

$$\Rightarrow m\ddot{r} + kr - kl_0 - mg = 0.$$

$$\Rightarrow m\overline{\ddot{(r - r_0)}} + k(r - r_0) = 0. \tag{4}$$

Following the same procedure for equation (2), we get

$$mr\ddot{\theta} + 2m\dot{r}\dot{\theta} + mg\sin\theta = 0,$$

$$\Rightarrow \left(mr\ddot{\theta}\right)\Big|_{\substack{r=r_0\\ \theta=\theta_0}} + \frac{\partial}{\partial r}(mr\ddot{\theta})\Big|_{\substack{r=r_0\\ \theta=0}} \cdot (r - r_0) + \frac{\partial}{\partial \ddot{\theta}}(mr\ddot{\theta})\Big|_{\substack{r=r_0\\ \ddot{\theta}=0}} \cdot (\ddot{\theta} - 0)$$

$$+(2m\dot{r}\dot{\theta})\Big|_{\substack{r=r_0\\ \theta=\theta_0}} + \frac{\partial}{\partial \dot{r}}(2m\dot{r}\dot{\theta})\Big|_{\substack{\dot{r}=r_0\\ \dot{\theta}=0}} \cdot (\dot{r} - 0) + \frac{\partial}{\partial \dot{\theta}}(2m\dot{r}\dot{\theta})\Big|_{\substack{\dot{r}=r_0\\ \dot{\theta}=0}} \cdot (\dot{\theta} - 0)$$

$$(mg\sin\theta)\Big|_{\substack{r=r_0\\ \theta=\theta_0}} + \frac{\partial}{\partial\theta}(mg\sin\theta)\Big|_{\substack{r=r_0\\ \theta=0}} \cdot (\theta - \theta_0) = 0,$$

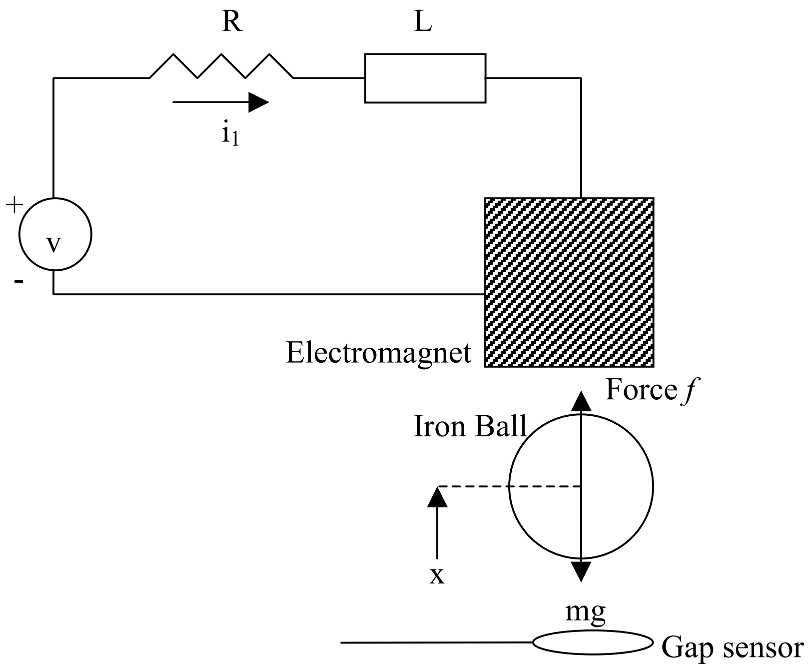
$$\Rightarrow mr_0\ddot{\theta} + mg\theta = 0. \tag{5}$$

Equations (4) and (5) represents the linearized differential equation of motion for the above system.

Example 2

Consider the electromagnetic suspension system shown in the figure. An electromagnet is located at the upper part of the experimental system. Utilizing the electromagnetic force f, we desire to suspend the iron ball. Note that this simple electromagnetic suspension system is essentially unworkable. Hence feedback control is indispensable. As a gap sensor, a standard induction probe of eddy current type is placed below the ball.

The electromagnet has an inductance 'L' and a resistance 'R'. Use the Taylor series approximation for the electromagnetic force. The current is $i_1 = (I_0 + i)$, where I_0 is the operating point and 'i' is the variable. The mass of the ball is 'm'. The gap $x_g = (X_0 + x)$, where X_0 is the operating point and 'x' is the variable. The electromagnetic force is given by $f = k\left(\frac{i_1}{x_g}\right)^2$, where 'k' is a constant. Determine the Linearized governing differential equations of motion.



From the figure, writing the governing differential equation for the electric circuit, we get

$$L\frac{di_1}{dt} + Ri_1 = v. \tag{6}$$

For the iron ball, writing the Newton's second law of motion, we have

$$m\ddot{x} = f - mg,$$

$$\Rightarrow m\ddot{x} = k\left(\frac{i_1}{x_g}\right)^2 - mg. \qquad (7)$$

Equations (6) and (7) represent the governing differential equations of motion for the electromagnetic suspension system. Note that equation (7) is non-linear in nature. In order to linearize the equations, the operating point has to be calculated.

Equating all the derivative terms in equation (6) and (7) to zero, we get

$$RI_0 = v,$$

$$k\left(\frac{I_0}{X_0}\right)^2 = mg,$$

where I_0 and X_0 are the operating points.

Therefore equation (6) reduces to

$$L\frac{di}{dt} + R(I_0 + i) = v,$$

$$\Rightarrow L\frac{di}{dt} + Ri = 0. \qquad (8)$$

Using the Taylor series expansion for equation (7), we get

$$m\ddot{x} = k\left(\frac{I_0}{X_0}\right)^2 + \frac{\partial}{\partial i_1}\left(k\left(\frac{i_1}{x_g}\right)^2\right)\Bigg|_{\substack{i_1=I_0\\ x_g=X_0}} \cdot (i_1 - I_0) + \frac{\partial}{\partial x_g}\left(k\left(\frac{i_1}{x_g}\right)^2\right)\Bigg|_{\substack{i_1=I_0\\ x_g=X_0}} \cdot (x_g - X_0) - mg,$$

$$\Rightarrow m\ddot{x} = \frac{2ki_1}{x_g}\frac{1}{x_g}\Bigg|_{\substack{i_1=I_0\\ x_g=X_0}} \cdot (i_1 - I_0) + \frac{2ki_1}{x_g}\left(-\frac{i_1}{(x_g)^2}\right)\Bigg|_{\substack{i_1=I_0\\ x_g=X_0}} \cdot (x_g - X_0),$$

$$\Rightarrow m\ddot{x} = \frac{2kI_0}{(X_0)^2}i - \frac{2k(I_0)^2}{(X_0)^3}x. \qquad (9)$$

Equations (8) and (9) represent the linearized differential equation for the system defined.

Assignment

Derive the equations of motion for the inverter pendulum problem and linearize the equations about the equilibrium point. (Do not assume small angle approximation.)