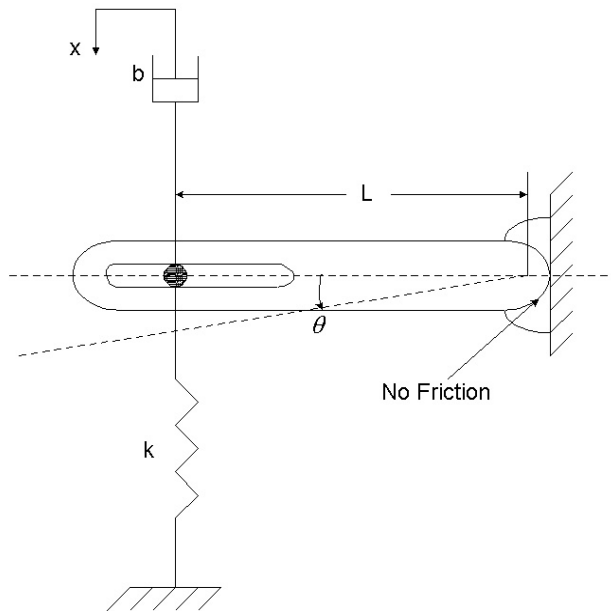


HANDOUT E.15 - EXAMPLES ON TRANSIENT RESPONSE OF FIRST AND SECOND ORDER SYSTEMS, SYSTEM DAMPING AND NATURAL FREQUENCY

Example 1

In the system shown below, $x(t)$ is the input displacement and $\theta(t)$ is the output angular displacement. Assume all masses involved are negligibly small and that all motions are restricted to be small. Obtain the response of the system for a unit step input. Assume zero initial conditions.



Writing the force balance equation for the above system, we get

$$\begin{aligned} b(\ddot{x} - L\ddot{\theta}) &= kL\theta, \\ L\ddot{\theta} + \frac{k}{b}L\theta &= \ddot{x}. \end{aligned} \quad (1)$$

Equation (1) represents the governing differential equation of motion.

Taking the Laplace transforms of equation (1), we get

$$\begin{aligned} \left(Ls + \frac{k}{b}L \right) \Theta(s) &= sX(s), \\ \Rightarrow \frac{\Theta(s)}{X(s)} &= \frac{1}{L} \frac{s}{s + (k/b)}. \end{aligned} \quad (2)$$

Equation (2) represents the transfer function of the system shown above. For a unit step input $X(s) = \frac{1}{s}$, therefore the output $\Theta(s)$ becomes,

$$\Theta(s) = \frac{1}{L} \frac{1}{s + (k/b)}. \quad (4)$$

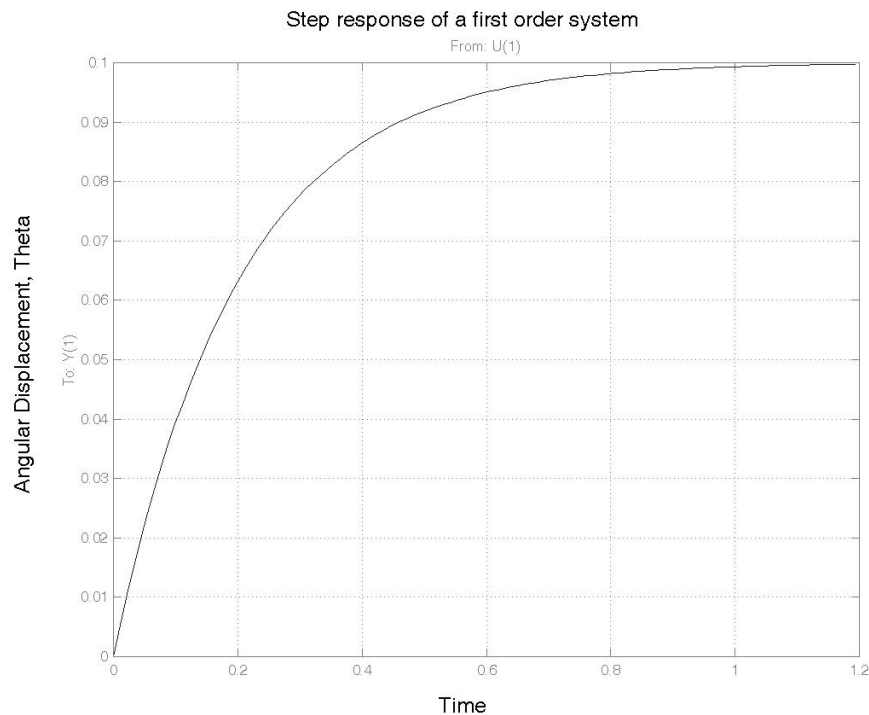
Taking the inverse Laplace transform of equation (4), we get

$$\theta(t) = \frac{1}{L} e^{-(k/b)t}. \quad (5)$$

Equation (5) represents the response of the system for a given step input. The MATLAB sequence to obtain the step response for a given 'L', 'k' and 'b' is given below.

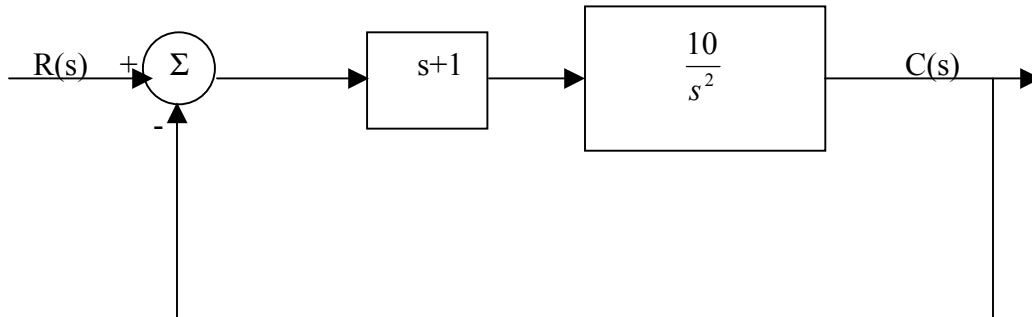
```
L = 2;  
k = 100;  
b = 20;  
  
num = 1/L;  
den = [1 k/b];  
  
sys = tf(num,den);  
step(sys)  
xlabel('Time')  
ylabel('Angular Displacement, Theta')  
title('Step response of a first order system')
```

The response of the system is as shown below.



Example 2

What is the unit step response of the system shown below?



The closed loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{10s + 10}{s^2 + 10s + 10}.$$

For a unit step input $R(s) = 1/s$, therefore,

$$\begin{aligned} C(s) &= \frac{10s + 10}{s^2 + 10s + 10} \cdot \frac{1}{s}, \\ \Rightarrow C(s) &= \frac{10s + 10}{(s + 5 + \sqrt{15})(s + 5 - \sqrt{15})s}, \\ \Rightarrow C(s) &= \frac{-4 - \sqrt{15}}{3 + \sqrt{15}} \frac{1}{s + 5 + \sqrt{15}} + \frac{-4 + \sqrt{15}}{3 - \sqrt{15}} \frac{1}{s + 5 - \sqrt{15}} + \frac{1}{s}. \end{aligned} \quad (6)$$

The inverse Laplace transform of the above equation yields,

$$\begin{aligned} c(t) &= \frac{4 + \sqrt{15}}{3 + \sqrt{15}} e^{-(5 + \sqrt{15})t} + \frac{4 - \sqrt{15}}{-3 + \sqrt{15}} e^{-(5 - \sqrt{15})t} + 1, \\ \Rightarrow c(t) &= -1.1455e^{-8.87t} + 0.1455e^{-1.13t} + 1. \end{aligned} \quad (7)$$

Equation (7) represents the response of the system to a unit step input.

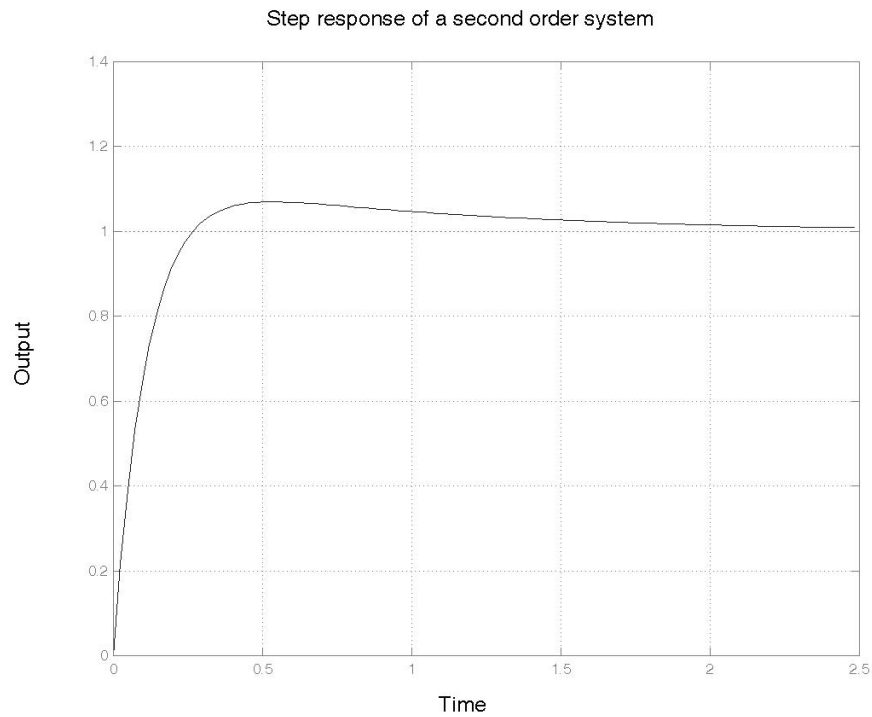
The MATLAB code for obtaining the unit step response of the above second order system is given below.

```
num = [10 10];
den = [1 10 10];

sys = tf(num,den);
step(sys)
```

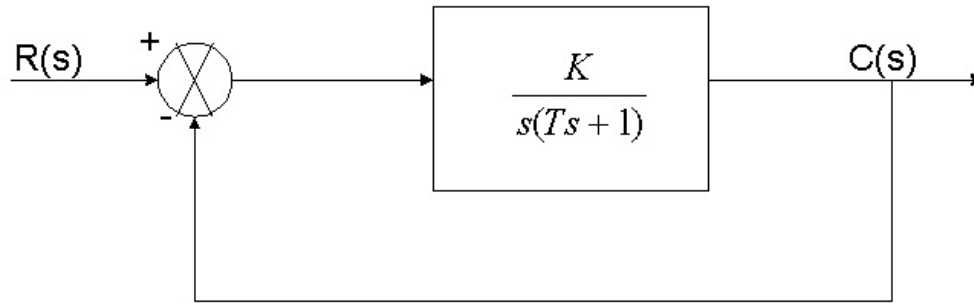
```
xlabel('Time')  
ylabel('Output')  
Title('Step response of a second order system')
```

The response is shown below. Note that there is an overshoot.

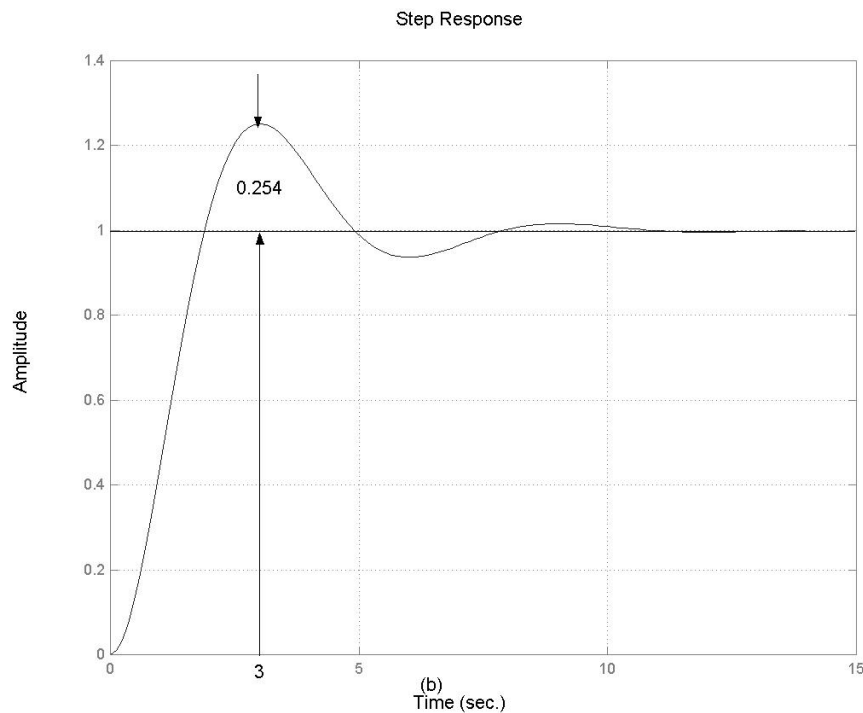


Example 3

When the system as shown in Figure (a) is subjected to a unit step input, the system output responds as shown in Figure (b). Determine the values of 'K' and 'T' from the response curve.



(a)



The maximum overshoot from the response curve is 25.4%. Therefore

$$M_p = 0.254,$$

$$\Rightarrow e^{-\pi \zeta / \sqrt{1-\zeta^2}} = 0.254,$$

$$\Rightarrow \zeta = 0.4.$$

From the response curve we have

$$t_p = 3,$$

$$\Rightarrow t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} = \frac{\pi}{\omega_n \sqrt{1-(0.4)^2}} = 3,$$

$$\Rightarrow \omega_n = 1.14.$$

From the block diagram, the closed loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{K}{Ts^2 + s + K}.$$

Hence

$$\omega_n = \sqrt{\frac{K}{T}}, \quad 2\zeta \omega_n = \frac{1}{T}.$$

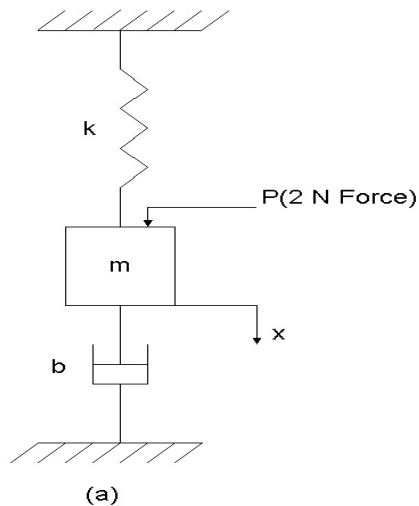
Therefore the values of 'K' and 'T' can be determined as

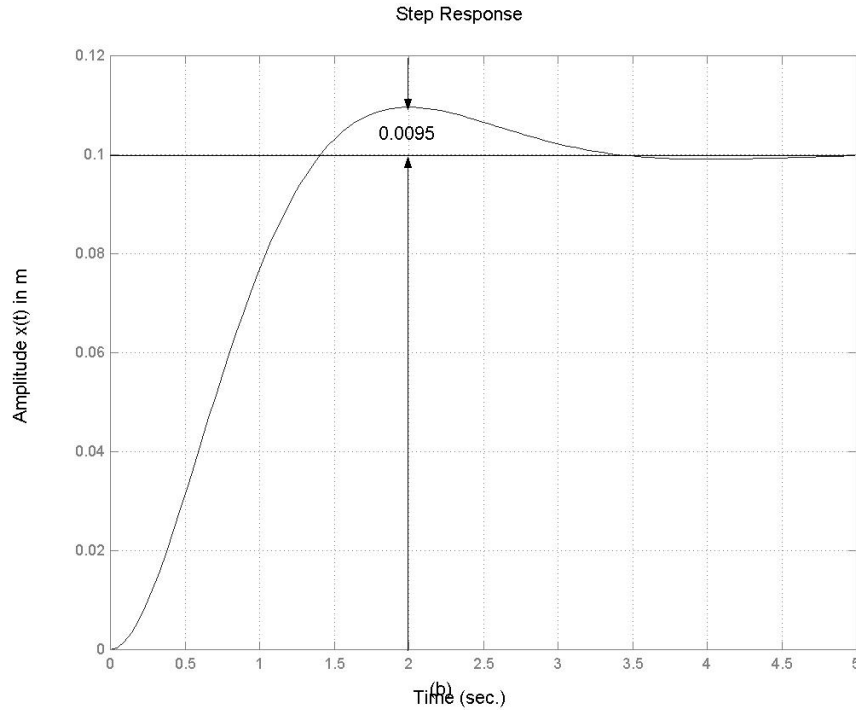
$$T = \frac{1}{2\zeta \omega_n} = \frac{1}{2 \times 0.4 \times 1.14} = 1.09,$$

$$K = \omega_n^2 T = (1.14)^2 \times 1.09 = 1.42.$$

Example 4

Figure (a) shows a mechanical vibratory system. When a 2 N force (step input) is applied to the system, the mass oscillates as shown in Figure (b). Determine 'm', 'b', and 'k' of the system from the response curve. The displacement 'x' is measured from the equilibrium position.





The transfer function of the system is

$$\frac{X(s)}{P(s)} = \frac{1}{ms^2 + bs + k}.$$

Since for step input of 2 lb, $P(s) = \frac{2}{s}$, we obtain

$$X(s) = \frac{2}{s(ms^2 + bs + k)}.$$

From the response curve, the steady state value of x is 0.1, hence from the final value theorem, we have

$$x(\infty) = \lim_{s \rightarrow 0} sX(s) = \lim_{s \rightarrow 0} \frac{2}{ms^2 + bs + k} = \frac{2}{k} = 0.1,$$

$$k = 20 \text{ N/m}.$$

From the response curve, the maximum overshoot is 0.0095, hence applying the formula for the maximum overshoot, we get

$$M_p = 0.0095,$$

$$\Rightarrow e^{-\frac{\pi \zeta}{\sqrt{1-\zeta^2}}} = 0.0095,$$

$$\Rightarrow \zeta = 0.6.$$

The peak time t_p is given by

$$t_p = 2,$$

$$\Rightarrow t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} = \frac{\pi}{\omega_n \sqrt{1-(0.6)^2}} = 2,$$

$$\Rightarrow \omega_n = 1.96 \text{ rad/sec}$$

Since,

$$\omega_n^2 = \frac{k}{m},$$

$$\Rightarrow m = \frac{k}{\omega_n^2} = \frac{20}{(1.96)^2} = 5.2 \text{ Kg.}$$

Then 'b' is determined as

$$2\zeta \omega_n = \frac{b}{m},$$

$$\Rightarrow b = 12.2 \text{ N-sec/m}$$

Example 5

Consider the second order system whose transfer function is given as

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2}.$$

For a unit step input, $R(s) = 1/s$, therefore the output is given by

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta \omega_n s + \omega_n^2)}.$$

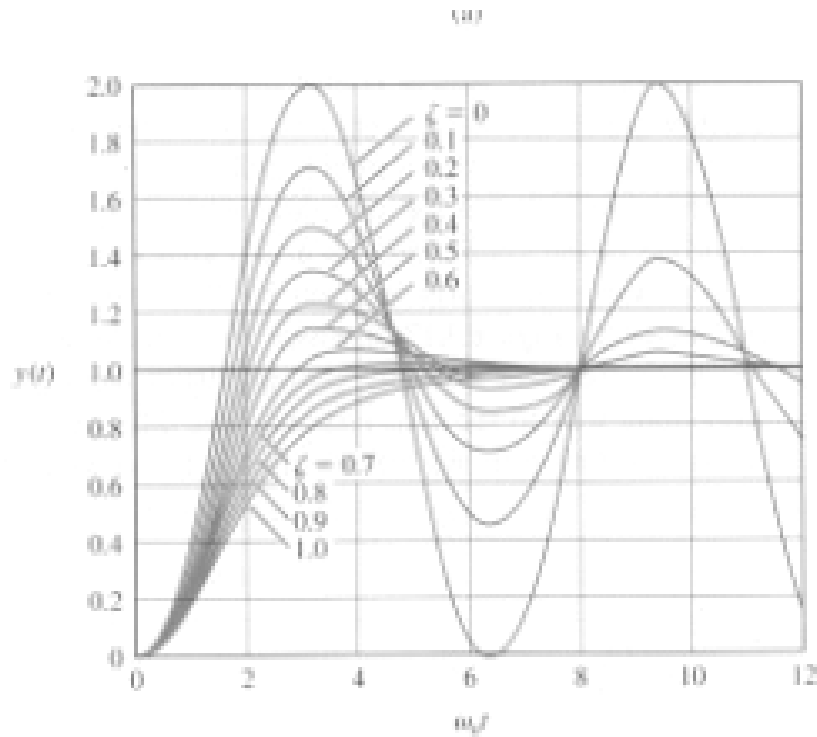
Expressing the above equation in partial fraction format, we have

$$C(s) = \frac{1}{s} - \frac{s + \zeta \omega_n}{(s + \zeta \omega_n)^2 + \omega_d^2} - \frac{\zeta \omega_n}{(s + \zeta \omega_n)^2 + \omega_d^2}.$$

Taking the inverse Laplace transform of the above equation and rearranging the terms, we get

$$c(t) = 1 - \frac{e^{-\zeta \omega_n t}}{\sqrt{1-\zeta^2}} \sin \left(\omega_d t + \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta} \right) \quad (8)$$

Equation (8) represents the generalized solution of a second order system. The following plot shows the step response of a second order system for various values of ζ .

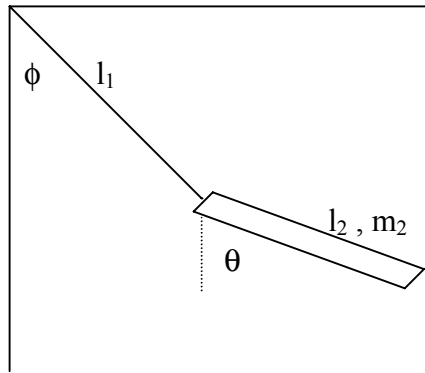


Example 6

Derive the governing differential equation of motion of a swinging bar supported at its ends by a cord. Solve the derived differential equation and plot the initial response of the system for the following initial conditions:

a) Arbitrary initial condition

Choose $l_1 = 1\text{m}$; $l_2 = 1\text{m}$; $m_2 = 5\text{Kg}$



The differential equations of motion for the above system when represented in a matrix form is

$$\begin{bmatrix} m_2 l_1 & \frac{m_2 l_2 \cos(\theta - \phi)}{2} \\ \frac{m_2 l_1 l_2 \cos(\theta - \phi)}{2} & \frac{m_2 l_2^2}{3} \end{bmatrix} \begin{Bmatrix} \ddot{\phi} \\ \ddot{\theta} \end{Bmatrix} = \begin{Bmatrix} -w_2 \sin \phi + \frac{m_2 l_2 \dot{\theta}^2 \sin(\theta - \phi)}{2} \\ -w_2 l_2 \sin \theta + \frac{m_2 l_1 l_2 \dot{\phi}^2 \sin(\theta - \phi)}{2} \end{Bmatrix}$$

Initial response

The second order differential equation has to be converted into a first order differential equation. Let

$$\phi = y(1);$$

$$\dot{\phi} = y(2);$$

$$\theta = y(3);$$

$$\dot{\theta} = y(4);$$

Substituting the above relations in the original nonlinear differential equation, we get the following nonlinear first order differential equation, which when represented in matrix form is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & m_2 l_1 & 0 & \frac{m_2 l_2 \cos(y(3) - y(1))}{2} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{m_2 l_1 l_2 \cos(y(3) - y(1))}{2} & 0 & \frac{m_2 l_2^2}{3} \end{bmatrix} \begin{bmatrix} y(1) \\ y(2) \\ y(3) \\ y(4) \end{bmatrix} = \begin{bmatrix} -w_2 \sin y(1) + \frac{y(2)}{2} \\ -w_2 \sin y(3) + \frac{y(4)}{2} \end{bmatrix}$$

MATLAB Code

In this type of a problem where the inertia matrix (mass matrix) is a function of the states or the variables, a separate M-file has to be written which incorporates a switch/case programming with a flag case of 'mass'.

For example if the differential equation is of the form,

$$M(t, y) * y'(t) = F(t, y),$$

then the right hand side of the above equation has to be stored in a separate m-file called 'F.m'. Similarly the inertia matrix (mass matrix) should also be stored in a separate m-file named 'M.m'. So, when the flag is set to the default, the function 'F.m' is called and later when the flag is set to 'mass' the function 'M.m' is called.

The code with the switch/case is given below. Note that it is a function file and should be saved as 'indmot_ode.m' in the current directory.

```
function varargout=indmot_ode(t,y,flag)

switch flag
case '' %no input flag
    varargout{1}=FF(t,y);
case 'mass' %flag of mass calls MM.m
    varargout{1}=MM(t,y);
otherwise
    error(['unknown flag ' flag '.']);
end
```

To store the right hand side of the original matrix form of differential equation, a separate function file must be written as shown below. Note that the name of the function is 'FF', so this file must be saved as 'FF.m'.

```
%the following function contains the right hand side of the
%differential equation of the form
%M(t,y)*y'=F(t,y)
%i.e. it contains F(t,y).it is also stored in a separate file named,
FF.m.

function yp=FF(t,y)
l1=1;
l2=1;
```

```
m2=5;  
g=9.81;  
w2=m2*g;  
yp=zeros(4,1);  
yp(1)=y(2);  
yp(2)=-w2*sin(y(1))+(m2*l2/2)*(y(4)^2)*sin(y(3)-y(1));  
yp(3)=y(4);  
yp(4)=(-w2*l2*sin(y(3)))/2+(m2*l1*l2/2)*(y(2)^2)*sin(y(3)-y(1));
```

Similarly, to store the mass matrix a separate function file is written which is stored as 'MM.m'.

```
% the following function contains the mass matrix.  
%it is separately stored in a file named, MM.m
```

```
function n = MM(t,y)  
l1=1;  
l2=1;  
m2=5;  
g=9.81;  
w2=m2*g;  
n1=[1 0 0 0];  
n2=[0 m2*l1 0 (m2*l2/2)*cos(y(3)-y(1))];  
n3=[0 0 1 0];  
n4=[0 (m2*l1*l2/2)*cos(y(3)-y(1)) 0 m2*l2*l2/3];  
n=[n1;n2;n3;n4];
```

To plot the response, the main file should call the function 'indmot_ode.m', which has the switch/case programming which in turn calls the corresponding functions depending on the value of the flag. For the main file to recognize the inertia matrix, the MATLAB command ODESET is used to set the mass to 'M (t, y)'.

```
tspan=[0 30]  
  
y0=[0.5233;0;1.0467;0] % Arbitrary Initial condition  
  
options=odeset('mass','M(t,y)')  
[t,y]=ode113('indmot_ode',tspan,y0,options)  
subplot(2,1,1)  
plot(t,y(:,1))  
grid  
xlabel('Time')  
ylabel('phi')  
  
subplot(2,1,2)  
plot(t,y(:,3))  
grid  
xlabel('Time')  
ylabel('Theta')
```

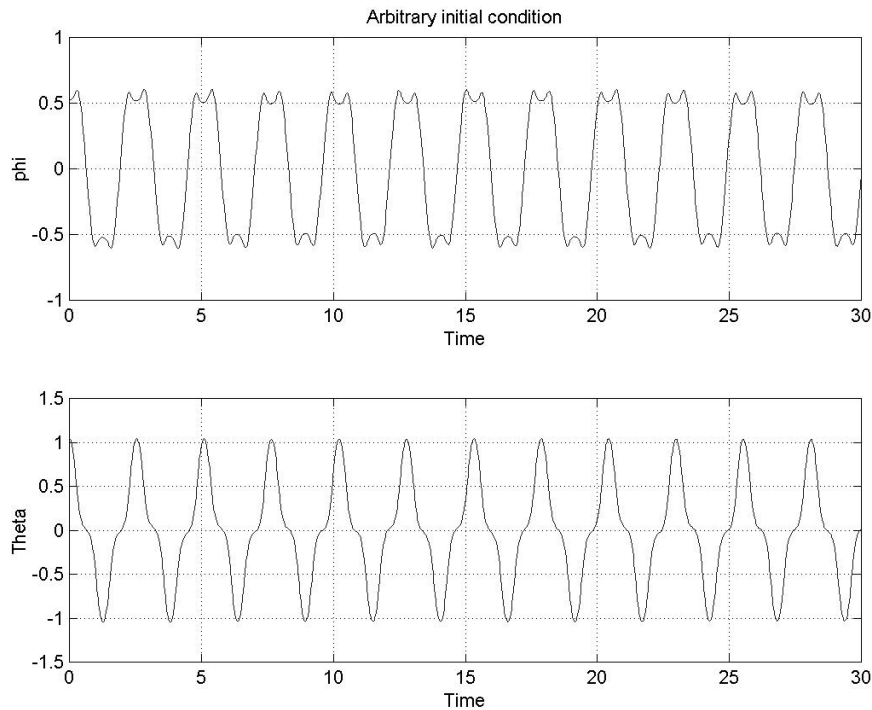
The above code plots the values of 'theta' and 'phi' with respect to time for the arbitrary initial condition case.

Notice the command “ subplot”.

subplot(m,n,p), breaks the Figure window into an m-by-n matrix of small axes and selects the p-th axes for the current plot. The axes are counted along the top row of the Figure window, then the second row, etc. For example,

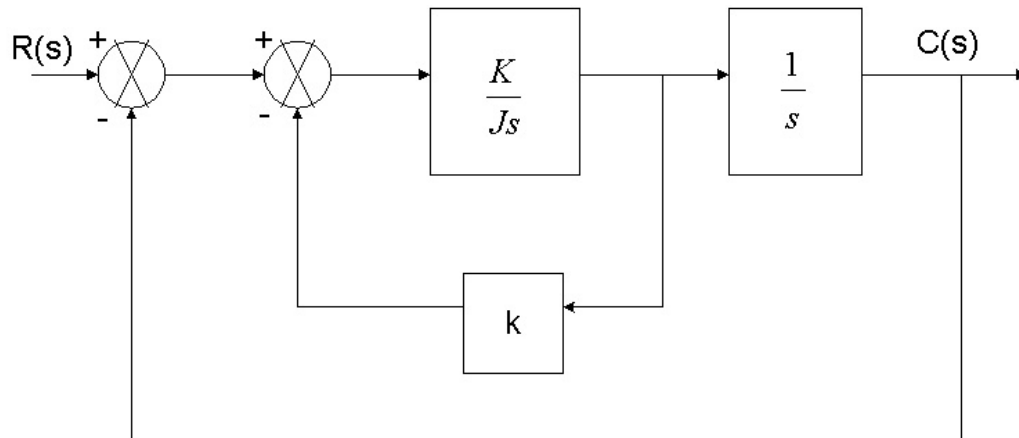
```
subplot(2,1,1), plot(income)
subplot(2,1,2), plot(outgo)
```

plots “income” on the top half of the window and “outgo” on the bottom half.



Assignment

1) Determine the values of 'K' and 'k' of the closed loop system shown below so that the maximum overshoot in unit step response is 25% and the peak time is 2 sec. Assume that $J = 1 \text{ Kg-m}^2$.



2) Consider the closed loop system given by

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2}.$$

Determine the values of ζ and ω_n so that the system responds to a step input with approximately 5% overshoot and with a settling time of 2 sec. (Use the 2% criterion).

3) Use MATLAB to plot the unit step response of the following system.

$$\frac{C(s)}{R(s)} = \frac{10}{s^2 + 2s + 10}, \text{ where } R(s) \text{ and } C(s) \text{ are the Laplace transforms of the input } r(t) \text{ and output } c(t) \text{ respectively.}$$