

HANDOUT E.24 - EXAMPLE HANDOUT ON STABILITY MARGINS AND COMPENSATION

Example 1:

Consider the system, whose open-loop transfer function is given by

$$G(s) = \frac{K}{s(0.2s + 1)(0.05s + 1)}.$$

Determine the value of K for a phase margin of 40° . For the value of K computed, determine the gain margin.

Sol:

The phase margin relation is given by

$$\begin{aligned} PM &= \angle G(j\omega) + 180, \\ \Rightarrow 40 &= -90 - \tan^{-1} \frac{0.2\omega}{1} - \tan^{-1} \frac{0.05\omega}{1} + 180, \\ \Rightarrow \tan^{-1} 0.2\omega + \tan^{-1} 0.05\omega &= 50, \\ \Rightarrow \tan^{-1} \left(\frac{0.25\omega}{1 - 0.01\omega^2} \right) &= 50, \\ \Rightarrow \omega &= 4 \text{ rad/sec} \end{aligned}$$

At this frequency, the magnitude must be equal to 1. Hence

$$\begin{aligned} |G(j\omega)|_{\omega=4} &= 1, \\ \Rightarrow \frac{K}{\omega \left(\sqrt{0.04\omega^2 + 1} \right) \left(\sqrt{0.0025\omega^2 + 1} \right)} \Big|_{\omega=4} &= 1, \\ \Rightarrow K &= 5.2. \end{aligned}$$

To determine the gain margin, first compute the frequency where the phase is equal to -180 degrees.

Therefore, we get

$$\begin{aligned} -90 - \tan^{-1} 0.2\omega - \tan^{-1} 0.05\omega &= -180, \\ \Rightarrow \tan^{-1} 0.2\omega + \tan^{-1} 0.05\omega &= 90, \end{aligned}$$

$$\Rightarrow \tan^{-1}\left(\frac{0.25\omega}{1-0.01\omega^2}\right) = 90,$$

$$\Rightarrow \frac{0.25\omega}{1-0.01\omega^2} \rightarrow \infty$$

Therefore the denominator must be equal to zero. Hence

$$1 - 0.01\omega^2 = 0,$$

$$\Rightarrow \omega^2 = \frac{1}{0.01},$$

$$\Rightarrow \omega = 10.$$

Substituting this value of frequency in the magnitude, we get

$$|G(j\omega)| = \frac{K}{\omega \left(\sqrt{0.04\omega^2 + 1} \right) \left(\sqrt{0.0025\omega^2 + 1} \right)} = \frac{5.2}{10(\sqrt{5})(\sqrt{1.25})} = 0.208.$$

Therefore the gain margin of the system is

$$GM = 0 - 20 \log(0.208) = 13.64 \text{ dB}.$$

Example 2:

Consider a type I unity feedback system with

$$G(s) = \frac{K}{s(s+1)}.$$

Design a Lead compensator so that $k_v = 12 \text{ sec}^{-1}$ and $\text{Pm} > 40^\circ$. Use MATLAB to verify that your design meets the specifications.

Sol:

The transfer function of the Lead Compensator is given by

$$D(s) = K_1 \frac{Ts + 1}{\alpha Ts + 1}, \text{ where } \alpha < 1.$$

For the design, consider K_1 always with the plant, such that the plant transfer function reduces to

$$G(s) = \frac{KK_1}{s(s+1)}.$$

Step 1: For the given steady state error constant, obtain the value of the gain.

$$k_v = \lim_{s \rightarrow 0} sG(s) = \lim_{s \rightarrow 0} \frac{KK_1}{(s+1)} = KK_1 = 12.$$

Therefore the plant transfer function reduces to

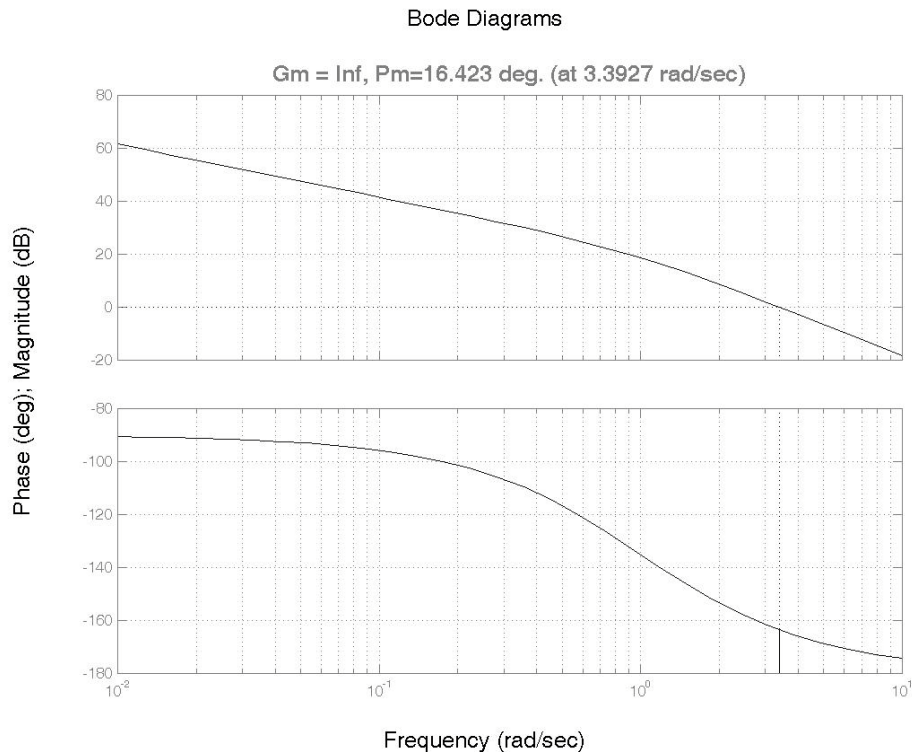
$$G(s) = \frac{12}{s(s+1)}. \quad (1)$$

Step 2: Plot the Bode plot of the transfer function given by equation (1) and compute the phase margin of the uncompensated system.

The Bode plot of the above system with $KK_1 = 12$ is given by following the sequence of MATLAB code given below.

```
s=tf('s');  
sys = 12/(s*(s+1));  
grid on;  
margin(sys)
```

The plot is given below.



Step 3: It can be seen that the phase margin of the uncompensated system is 16.4 degrees. The required phase margin should be greater than 40 degrees. Let us design for 42 degrees. Therefore the phase increase that is to be provided by the lead compensator is

$$\phi_1 = 42 - 16.4 + 7 = 32.6 \approx 33.$$

Note that 7° is used as a margin of safety.

Put $\phi_1 = \phi_{\max}$ and using the relation $\alpha = \frac{1 - \sin \phi_{\max}}{1 + \sin \phi_{\max}}$, obtain the value of α .

Therefore,

$$\alpha = \frac{1 - \sin(33)}{1 + \sin(33)} = 0.2948.$$

Step 4: Using the value of α calculated, compute the frequency where the magnitude is equal to $-10\log(1/\alpha)$.

$$\begin{aligned} |G(j\omega)| &= -10\log\left(\frac{1}{\alpha}\right). \\ \Rightarrow 20\log\frac{12}{\omega\sqrt{\omega^2 + 1}} &= -10\log\left(\frac{1}{0.2948}\right), \\ \Rightarrow \omega &\approx 6.23. \end{aligned}$$

Step 5: Choose the above frequency as the maximum frequency given by the relation

$$\omega_{\max} = \frac{1}{T\sqrt{\alpha}}.$$

Therefore

$$\begin{aligned} T &= \frac{1}{\omega_{\max}\sqrt{\alpha}}, \\ T &= \frac{1}{(6.23)(\sqrt{0.2948})}, \\ \Rightarrow T &= 0.2956. \end{aligned}$$

Therefore the Lead compensator is given by the transfer function,

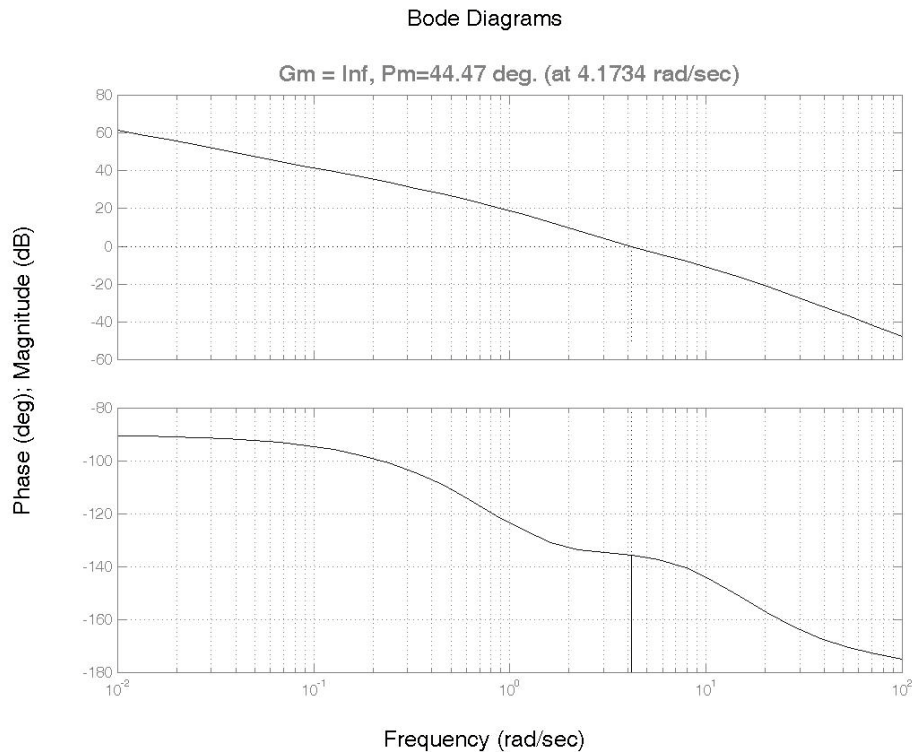
$$D(s) = \frac{(Ts + 1)}{(\alpha Ts + 1)} = \frac{(0.2956s + 1)}{(0.0871s + 1)}.$$

(Note that K1 is not considered in this transfer function, as it had already been accounted for in the plant transfer function.)

Plotting the Bode plot of the compensated system using the following sequence, we get

```
s=tf('s');
sys = 12/(s*(s+1));
comp = (0.2956*s+1)/(0.0871*s+1);
grid on;
margin(sys*comp)
```

From the bode plot it can be seen that the phase margin is 44.47 degrees. Hence the design specifications have been satisfied.



Example 3:

For the system with open-loop transfer function

$$G(s) = \frac{K}{s \left(\frac{s}{1.4} + 1 \right) \left(\frac{s}{3} + 1 \right)}$$

design a lag compensator such that the phase margin is approximately equal to 40° and the steady state velocity error constant is 10 sec⁻¹.

Sol:

The transfer function of the Lag compensator is given by

$$D(s) = \alpha \frac{Ts + 1}{\alpha Ts + 1}.$$

For the case of design, the gain α is always considered with the plant transfer function so that the plant transfer function reduces to

$$G(s) = \frac{K\alpha}{s\left(\frac{s}{1.4} + 1\right)\left(\frac{s}{3} + 1\right)}.$$

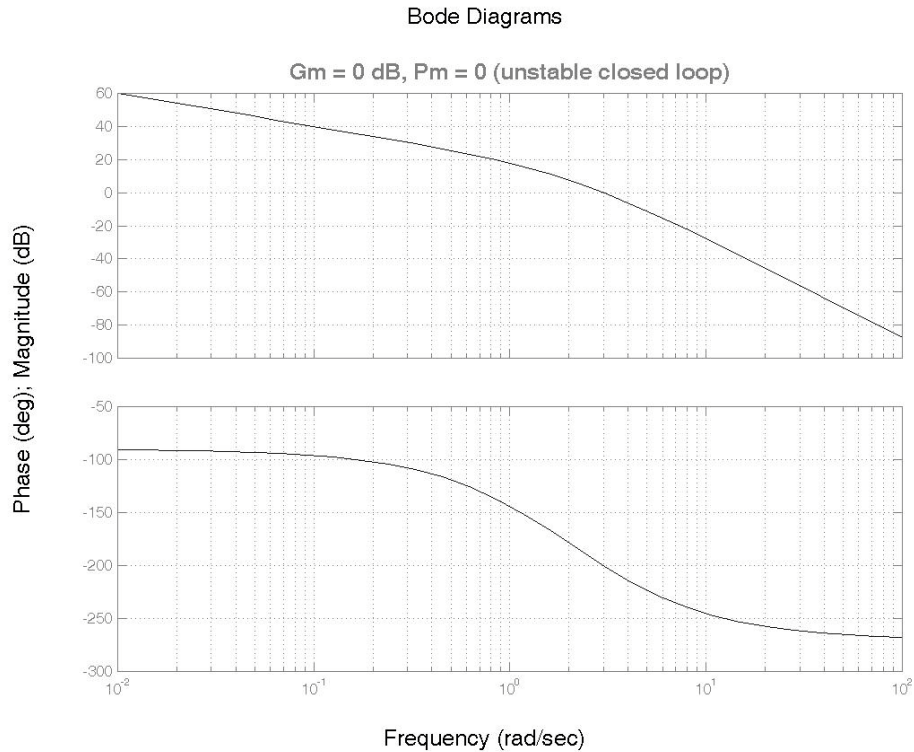
Step 1: For the given steady state error constant, obtain the value of $K\alpha$.

$$k_v = \lim_{s \rightarrow 0} sG(s) = \lim_{s \rightarrow 0} \frac{K\alpha}{\left(\frac{s}{1.4} + 1\right)\left(\frac{s}{3} + 1\right)} = K\alpha = 10.$$

Therefore the plant transfer function reduces to

$$G(s) = \frac{10}{s\left(\frac{s}{1.4} + 1\right)\left(\frac{s}{3} + 1\right)}.$$

Step 2: Since the specified phase margin is 40 degrees, let us design the compensator such that the final phase margin of the system is 45 degrees, assuming a margin of safety of 5 degrees. Compute the frequency at which the Bode plot of the system gives a phase margin of 45 degrees. The bode plot is given below.



From the Bode plot, the frequency ω_{c2} at which the phase margin is 45 degrees is 0.807 rad/sec, i.e., $\omega_{c2} = 0.807$ rad/sec.

Step 3: Using the frequency computed, compute the magnitude of the system at that frequency and equate it to $20\log(\alpha)$. Therefore

$$\left| G(j\omega) \right|_{\omega=0.701} = 20\log(\alpha),$$

$$\Rightarrow \alpha = 10.35.$$

Step 4:

Choose,

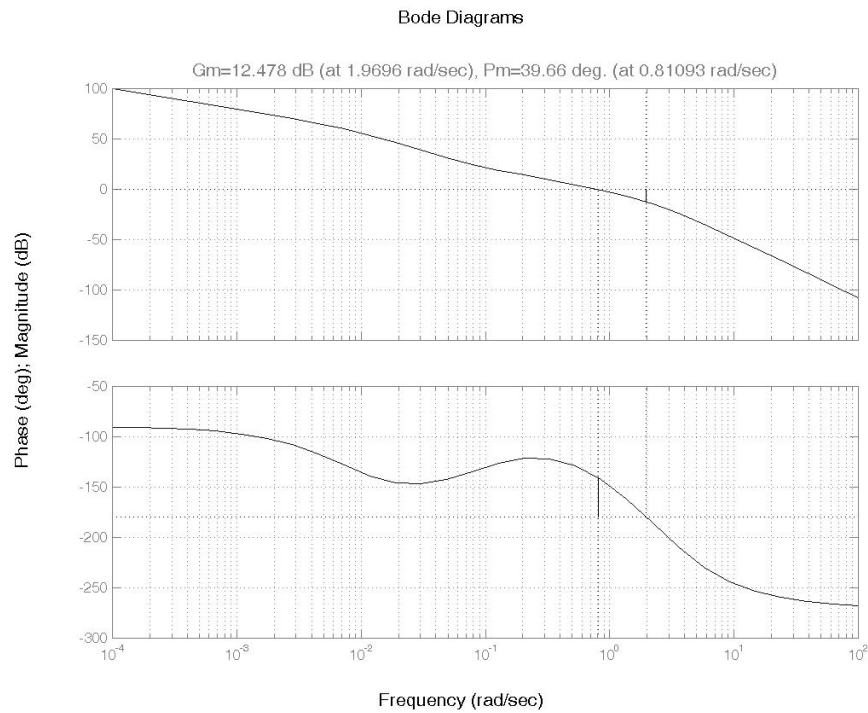
$$\frac{1}{T} = \frac{\omega_{c2}}{10} = \frac{0.807}{10} = 0.0807,$$

$$\Rightarrow T = 12.39.$$

Therefore the transfer function of the Lag compensator is given by

$$D(s) = \frac{(12.39s + 1)}{(128.253s + 1)}.$$

The Bode plot of the compensated system is given below.



From the above Bode plot, it can be seen that the phase margin of the above system is approximately equal to 39.66 degrees. Hence the design specifications are satisfied.

Recommended Reading

“Feedback Control of Dynamic Systems” Fourth Edition, by Gene F. Franklin et.al – pp 417 – 442.

Recommended Assignment

“Feedback Control of Dynamic Systems” Fourth Edition, by Gene F. Franklin et.al – problems 6.45, 6.48.