HANDOUT E.3 - EXAMPLES ON LAPLACE TRANSFORMS

Example 1

Evaluate the Laplace transform of the following functions.

a) $f(t) = e^{-\alpha t}$.

α is a positive real number.

The Laplace transform of a function *f*(t) is given by

$$L[f(t)] = \int_0^\infty f(t) e^{-st}\, dt.$$

Therefore,

$$L[f(t)] = \int_0^\infty e^{-\alpha t} e^{-st}\, dt = \int_0^\infty e^{-(s+\alpha)t}\, dt$$

$$= -\left.\frac{e^{-(s+\alpha)t}}{(s+\alpha)}\right|_0^\infty = \frac{1}{s+\alpha}.$$

b) $f(t) = \cos\omega\, t.$

Here ω is a positive real number.

$$L[f(t)] = \int_0^\infty \cos\omega\, t\; e^{-st}\, dt$$

Expressing $\cos\omega\, t$ in exponential form gives

$$\cos\omega\, t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}.$$

Then,

$$L[\cos\omega\ t] = \frac{1}{2}\left(\int_0^\infty e^{j\omega\ t}e^{-st}dt + \int_0^\infty e^{-j\omega\ t}e^{st}dt\right)$$

$$= \frac{1}{2}\left(\int_0^\infty e^{(j\omega\ -s)t}dt + \int_0^\infty e^{(-j\omega\ t-s)t}dt\right)$$

$$= \frac{1}{2}\left[\frac{e^{(j\omega\ -s)t}}{(j\omega\ -s)} + \frac{e^{(-j\omega\ -s)t}}{(-j\omega\ -s)}\right]_0^\infty$$

$$= \frac{1}{2}\left(-\frac{1}{(j\omega\ -s)} - \frac{1}{(-j\omega\ -s)}\right) = \frac{s}{s^2+\omega^2}.$$

c) $f(t) = te^{-\alpha t}$.

Using the theorem, which states that

$L[tf(t)] = -\dfrac{d}{ds}F(s)$, we get

$$L[te^{-\alpha t}] = -\frac{d}{ds}(L[e^{-\alpha t}]) = -\frac{d}{ds}\left(\frac{1}{s+\alpha}\right) = \frac{1}{(s+\alpha)^2}.$$

d) $f(t) = e^{-\alpha t}\cos\omega\ t.$

Using the theorem, which states that

$L[e^{at}f(t)] = F(s-a),$ we get

$$L[e^{-\alpha t}\cos\omega\ t] = \frac{(s+\alpha)}{(s+\alpha)^2+\omega^2}.$$

Example 2

a) Determine the inverse Laplace transform of the following function

$$F(s) = \frac{3}{(s^2+3s-10)}$$

Simplifying F(s) using partial fractions, we have

$$F(s)=\frac{3}{(s^2+3s-10)}=\frac{3}{(s+5)(s-2)}=\frac{A}{s+5}+\frac{B}{s-2}.$$

$$A=(s+5)F(s)\big|_{s=-5}=\left.\frac{3}{(s-2)}\right|_{s=-5}=-\frac{3}{7},$$

$$B=(s-2)F(s)\big|_{s=2}=\left.\frac{3}{(s+5)}\right|_{s=2}=\frac{3}{7}.$$

Therefore,

$$F(s)=\frac{-(3/7)}{s+5}+\frac{(3/7)}{s-2}.$$

Therefore,

$$L^{-1}[F(s)]=L^{-1}\left\{\frac{-(3/7)}{s+5}+\frac{3/7}{s-2}\right\}$$

$$=-\frac{3}{7}L^{-1}\left\{\frac{1}{s+5}\right\}+\frac{3}{7}L^{-1}\left\{\frac{1}{s-2}\right\}$$

$$=-\frac{3}{7}e^{-5t}+\frac{3}{7}e^{2t}.$$

b) Determine the inverse Laplace transform of the function,

$$F(s)=\frac{s+3}{(s+1)(s+2)^2}.$$

We write the partial fraction as

$$F(s)=\frac{A}{(s+1)}+\frac{B}{(s+2)}+\frac{C}{(s+2)^2}.$$

Then,

$$A=(s+1)F(s)\big|_{s=-1}=\left.\frac{(s+3)}{(s+2)^2}\right|_{s=-1}=2,$$

$$B = \frac{d}{ds}(s+2)^2 F(s)\Big|_{s=-2} = \frac{d}{ds}\frac{(s+3)}{(s+1)}\Bigg|_{s=-2} = -2,$$

$$C = (s+2)^2 F(s)\Big|_{s=-2} = \frac{(s+3)}{(s+1)}\Bigg|_{s=-2} = -1.$$

Therefore,

$$F(s) = \frac{2}{(s+1)} - \frac{2}{(s+2)} - \frac{1}{(s+2)^2},$$
$$\Rightarrow f(t) = (2e^{-t} - 2e^{-2t} - te^{-2t}).$$

Assignment

1) Determine the Laplace transform of the following functions.

$a)$ $f(t) = -10e^{-5t}$,

$b)$ $h(t) = \dfrac{1}{t+2}$,

$c)$ $g(t) = 6t + e^t \cos t$.

2) Determine the inverse Laplace transform of the following functions.

$a)$ $F(s) = \dfrac{5}{(3s+2)(s+1)}$,

$b)$ $G(s) = \dfrac{(2s-1)}{(s^2-6s+5)}$,

$c)$ $H(s) = \dfrac{1}{(s^2+s)}$.

Recommended Reading

"Feedback Control of Dynamic Systems" Fourth Edition, by Gene F. Franklin et.al – pp 96 - 115.

Recommended Assignment

"Feedback Control of Dynamic Systems" Fourth Edition, by Gene F. Franklin et.al – problems 3.2a, 3.2c, 3.3b, 3.5b, 3.7 a-d.