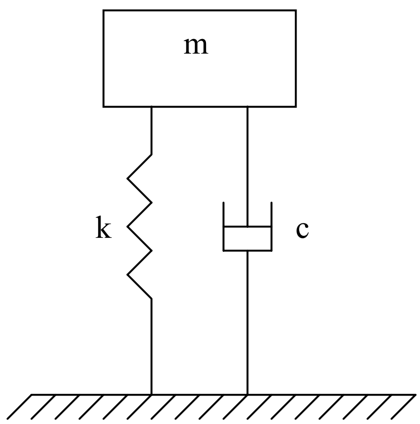
HANDOUT E.5 - EXAMPLES ON MODELLING OF TRANSLATIONAL MECHANICAL SYSTEMS

A generalized procedure has been followed in all the examples in this handout to derive the governing differential equations of motion. Note that the time dependence of all variables is ignored for all manipulations.

Example 1: One DOF system

Consider the system shown below.



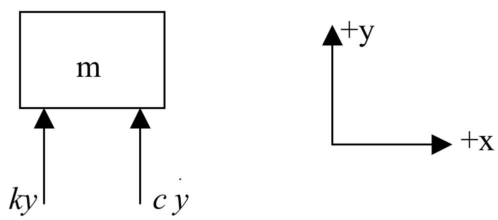
Kinematics stage

In this stage the position, velocity and the acceleration of all the rigid bodies in the system are defined. In the above system, there is only one rigid body. Let the displacement moved by the body from the static equilibrium position be equal to 'y' units. Then the velocity and the acceleration of the body is defined as $\dot{y}$ and $\ddot{y}$ units respectively. This completes the kinematics stage.

Kinetics stage

In this stage the Newton's second law of motion, which states that the sum of all the forces acting on the body is equal to the product of its mass and its acceleration, is applied to obtain the final governing differential equation of motion.

Free body diagram of the body of mass 'm'



Note that the gravity force is not considered in the free body diagram, as the displacement of the body is considered from the static equilibrium position. Hence the spring force due to the initial compression of the spring balances the gravity force.

Writing the Newton's second law of motion, we have

$$\sum F_y = ma$$

$$\Rightarrow ky + c\dot{y} = -m\ddot{y},$$

$$m\ddot{y} + c\dot{y} + ky = 0. \tag{1}$$

The block does not move in the x-direction.

Equation (1) represents the governing differential equation of motion of the above-defined system.

Equation (1) can be rewritten as

$$\ddot{y} + \frac{c}{m}\dot{y} + \frac{k}{m}y = 0. \tag{2}$$

The generalized second order differential equation is given by

$$\ddot{y} + 2\zeta\,\omega_n\,\dot{y} + \omega_n^2 = 0, \tag{3}$$

where ζ is the damping ratio and ω_n is the natural frequency of the system.

Comparing equations (2) and (3), we have

$$\frac{c}{m} = 2\zeta\,\omega_n,$$

$$\frac{k}{m} = \omega_n^2.$$

State-space representation

Let the states of the system be defined as

$$y = x_1,$$

$$\dot{y} = x_2. \tag{4}$$

From the above relations it can be concluded that

$$\dot{x}_1 = x_2. \tag{5}$$

Substituting the relations given by equation (4) in equation (2), we have

$$\ddot{y} + \frac{c}{m}\dot{y} + \frac{k}{m}y = 0$$
$$\Rightarrow \dot{x}_2 + \frac{c}{m}x_2 + \frac{k}{m}x_1 = 0,$$
$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{c}{m}x_2. \tag{6}$$

Rewriting equations (5) and (6) in matrix format, we get

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{c}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{7}$$

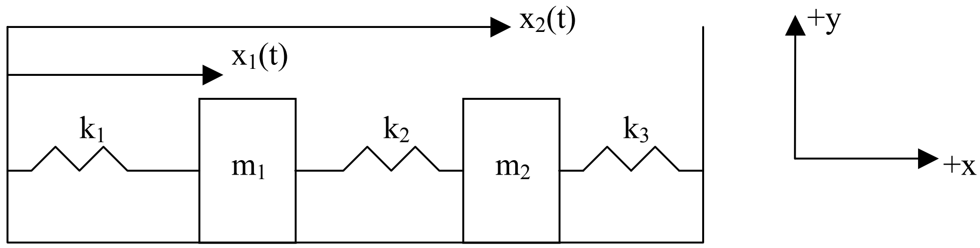
If the output of the system is the displacement of the block , then the output equation can be written in the matrix form as follows.

$$Y = y = x_1,$$
$$Y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{8}$$

Equations (7) and (8) represent the state-space representation of the system defined.

Example 2: Two DOF system

Consider the system given below.



Note: The time dependence is ignored for all future manipulations.

Kinematics stage

In this stage, the position, velocity and the acceleration of all the rigid bodies are defined. From the above figure, it can be seen that there are two rigid bodies. The total number of degrees of freedom of the system is two. The degrees of freedom of the system are defined as the horizontal displacement of the two bodies of mass 'm_1' and 'm_2'. Let the displacement of the bodies of mass 'm_1' and 'm_2' be equal to 'x_1' and 'x_2' respectively.

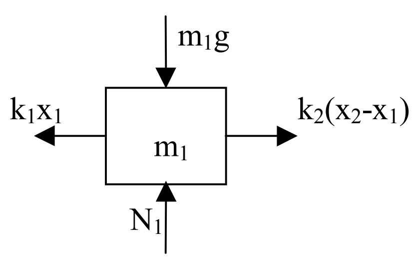
The velocity and acceleration of the body with mass 'm_1' is given by $\dot{x}_1$ and $\ddot{x}_1$ respectively. Similarly the velocity and the acceleration of the body with mass 'm_2' is $\dot{x}_2$ and $\ddot{x}_2$ respectively. This completes the kinematics stage.

Kinetics stage

In this stage, the Newton's second law of motion is used to obtain the final governing differential equation of motion. To write the force or the torque balance equations, we need to draw the free body diagram of each rigid body.

Assume x_2 to be greater than x_1.

Free body diagram of body of mass 'm_1'



Writing the Newton's second law of motion, which states that the sum of the forces acting on the body must be equal to the product of its mass and acceleration.

$$\sum F_x = ma$$

$$\Rightarrow k_2(x_2 - x_1) - k_1 x_1 = m_1 \ddot{x}_1,$$

$$m_1 \ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = 0. \quad (9)$$

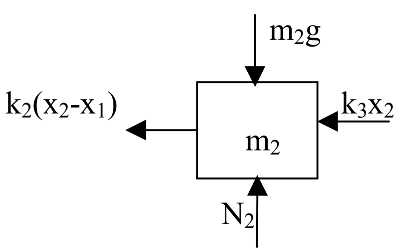
Similarly

$$\sum F_y = ma$$
$$\Rightarrow N_1 - m_1 g = 0,$$

where N_1 is the reaction force of the ground on the block.

Note that the block does not move in the y-direction and hence does not have any acceleration in that direction.

Free body diagram of body of mass 'm_2'



Writing the Newton's second law of motion for this body, we have

$$\sum F_x = ma$$

$$\Rightarrow -k_2(x_2 - x_1) - k_3 x_2 = m_2 \ddot{x}_2,$$

$$m_2 \ddot{x}_2 - k_2 x_1 + (k_2 + k_3)x_2 = 0. \quad (10)$$

$$\sum F_y = ma$$
$$\Rightarrow N_2 - m_2 g = 0,$$

where N_2 is the reaction force of the ground on the block of mass 'm_2'.

Even in this case the block does not move in the y-direction and has no acceleration in that direction.

Equations (9) and (10), represent the governing equation of motion for the system defined. Rewriting the equations in the form of a matrix, we have

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} (k_1+k_2) & -k_2 \\ -k_2 & (k_2+k_3) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{11}$$

The above equation is of the form

$$[\mathbf{M}]\left[\ddot{\mathbf{X}}\right] + [\mathbf{K}][\mathbf{X}] = 0$$

where [**M**] is the mass matrix, [**K**] is the stiffness matrix and [**X**] is the vector containing the displacements of the blocks.

To calculate the Eigen values and Eigen vectors

Let the constants in the above system be defined as follows

$m_1 = 3$ Kg,

$m_2 = 1.5$ Kg,

$k_1 = 2000$ N/m,

$k_2 = 1000$ N/m,

$k_3 = 3000$ N/m.

Therefore the mass and the stiffness matrices are

$$\mathbf{M} = \begin{bmatrix} 3 & 0 \\ 0 & 1.5 \end{bmatrix},$$

$$\mathbf{K} = \begin{bmatrix} 3000 & -1000 \\ -1000 & 4000 \end{bmatrix}.$$

To obtain the Eigen values and Eigen vectors of the above system, use the following MATLAB code.

```
% this code calculates the eigen values and eigen
% vectors associated with the defined system.

m = [3 0;0 1.5];
k = [3000 -1000;-1000 4000];
[v,d] = eig(k,m)
wnat = sqrt(d)
```

The result of the above code is

```
v =

    0.9372   -0.1830
    0.3489    0.9831


d =

  1.0e+003 *

    0.8759         0
         0    2.7908


wnat =

   29.5957         0
         0   52.8276
```

The diagonal elements in the matrix 'd' represents the Eigen values of the system and the corresponding column vector in the matrix 'v' represents the Eigen vector associated with that particular Eigen value.

Also note that the natural frequency of the system is defined as the square root of the Eigen values. The vector 'wnat' gives the natural frequency of the system. The physical significance of the Eigen values and Eigen vectors is explained in detail in the handout on Eigen values and Eigen vectors.

State-space representation

Let the states of the system be defined as

$$\begin{aligned} x_1 &= X_1, \\ \dot{x}_1 &= X_2, \\ x_2 &= X_3, \\ \dot{x}_2 &= X_4. \end{aligned} \tag{12}$$

From the above relations the following equations can be derived

$$\begin{aligned} \dot{X}_1 &= X_2, \\ \dot{X}_3 &= X_4. \end{aligned} \tag{13}$$

Substituting the relations given by equation (12) in equation (9), we get

$$m_1 \ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = 0$$

$$\Rightarrow m_1 \dot{X}_2 + (k_1 + k_2)X_1 - k_2 X_3 = 0,$$

$$\dot{X}_2 = -\frac{(k_1 + k_2)}{m_1} X_1 + \frac{k_2}{m_1} X_3. \tag{14}$$

Similarly substituting the relations given by equation (12) in equation (10), we get

$$m_2 \ddot{x}_2 - k_2 x_1 + (k_2 + k_3)x_2 = 0$$

$$\Rightarrow m_2 \dot{X}_4 - k_2 X_1 + (k_2 + k_3)X_3 = 0,$$

$$\dot{X}_4 = \frac{k_2}{m_2} X_1 - \frac{(k_2 + k_3)}{m_2} X_3. \tag{15}$$

Rewriting equations (13), (14) and (15) in matrix format, we get

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \\ \dot{X}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{(k_1 + k_2)}{m_1} & 0 & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & -\frac{(k_2 + k_3)}{m_2} & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} \tag{16}$$

If the output of the system is the displacement of the block of mass 'm_2' then the output equation can be written in the matrix format as

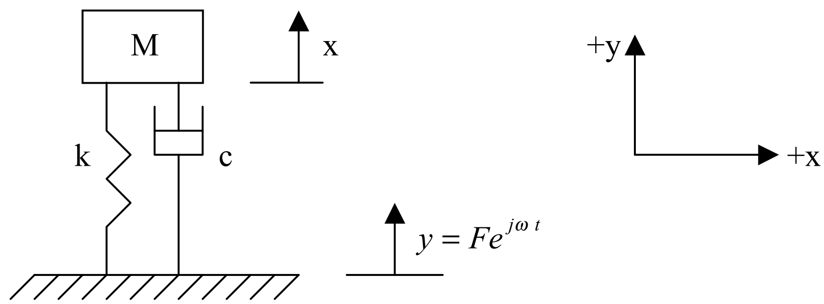
$$Y = x_2 = X_3,$$

$$\Rightarrow Y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}. \tag{17}$$

Equations (16) and (17) represent the state-space form of the above-defined system.

Example 3: Base excitation problem

Consider the system shown below, in which the base of the system moves.



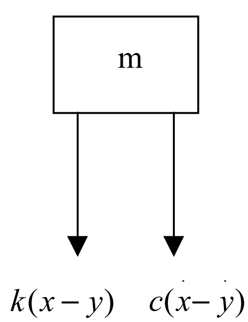
Notice that, the base of the system moves by 'y' units independent of the mass. But in this case, we however know how the base moves, as given by the harmonic function. So in effect this is just a one degree of freedom problem. That is in other words, there are two degrees of freedom for this system, but out of which one degree of freedom is known.

Kinematics stage

The position, velocity and the acceleration of the mass are $x,\ \dot{x},\ \ddot{x}$ respectively. This completes the kinematics stage.

Kinetics stage

Free body diagram of the mass



Note that the displacements 'x' and 'y' in the system are from the static equilibrium position. Hence the spring force due to the initial compression of the spring will balance the gravity force.

Writing the Newton's force balance equation, we have

$$\sum F = ma$$

$$\Rightarrow -k(x-y) - c(\dot{x} - \dot{y}) = m\ddot{x},$$

$$\Rightarrow m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky. \tag{18}$$

Since,

$$y = Fe^{j\omega t},$$

$$\Rightarrow \dot{y} = Fj\omega\, e^{j\omega t}.$$

Substituting the above relations in equation (18), we get

$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky,$$

$$\Rightarrow m\ddot{x} + c\dot{x} + kx = c(Fj\omega\, e^{j\omega t}) + k(Fe^{j\omega t}),$$

$$\Rightarrow m\ddot{x} + c\dot{x} + kx = F(k + jc\omega)e^{j\omega t}. \tag{19}$$

Equation (19) represents the governing differential equation of motion for the system in which the base is excited with a known amplitude and phase. This equation is similar to that of a forced one degree of freedom system.

State-space representation

Let the states of the system be

$$x = x_1, \qquad \dot{x} = x_2. \tag{20}$$

From the above relations it cab be concluded that

$$\dot{x}_1 = x_2. \tag{21}$$

Substituting the relations given by equation (20) in equation (19), we have

$$m\ddot{x} + c\dot{x} + kx = F(k + jc\omega)e^{j\omega t},$$

$$\Rightarrow m\dot{x}_2 + cx_2 + kx_1 = F_1,$$

where

$$F_1 = F(k + jc\omega)e^{j\omega t}.$$

$$\Rightarrow \dot{x}_2 = \frac{F_1}{m} - \frac{k}{m}x_1 - \frac{c}{m}x_2. \tag{22}$$

Combining equations (21) and (22) in a matrix format, we get

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} F_1. \tag{23}$$

If the output of the system is the velocity of the mass, then the output equation can be written in the matrix format as

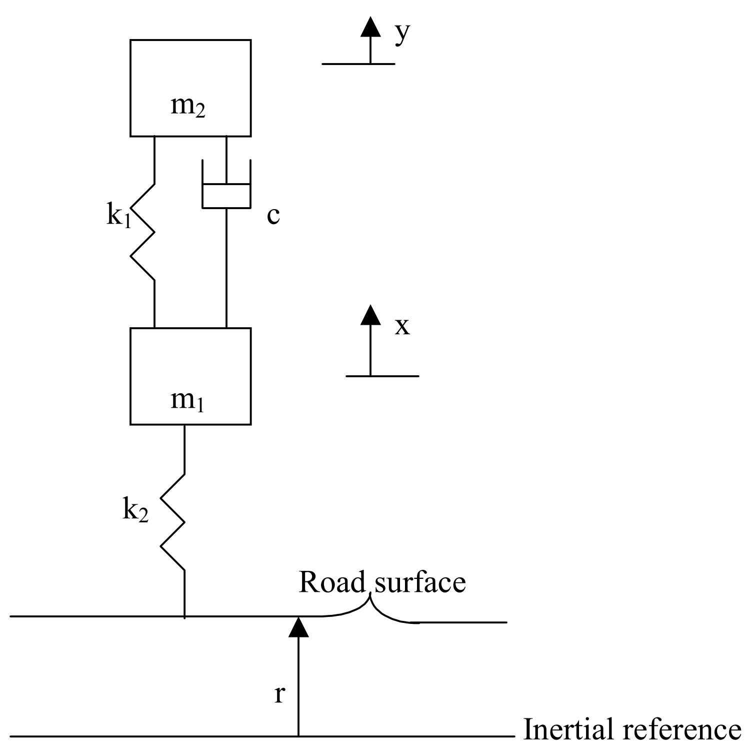
$$Y = \dot{x} = x_2,$$

$$\Rightarrow Y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{24}$$

Equations (23) and (24) represent the state-space form of the system defined.

Example 4: The quarter-car model

Consider the system shown below.



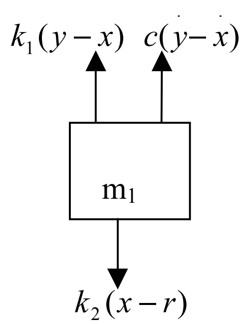
The system shown below is an approximation of a suspension model for one wheel of an automobile. The displacements of the masses, ‘x’ and ‘y’ are from their equilibrium positions.

Kinematics stage

The velocity and acceleration of the two masses are given as $\dot{x},\ \ddot{x},\ \dot{y},\ \ddot{y}$ respectively.

Kinetics stage

Free body diagram of the body of mass m_1



Note that the gravity force is neglected, as the spring forces due to the initial compression of the springs balance the gravity force.

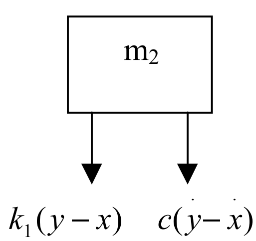
Writing the Newton’s second law of motion, we get

$$\sum F = ma,$$

$$\Rightarrow k_1(y-x)+c(\dot{y}-\dot{x})-k_2(x-r)=m_1\ddot{x},$$

$$\Rightarrow m_1\ddot{x}+(k_1+k_2)x-k_1y-c(\dot{y}-\dot{x})=k_2r. \qquad (25)$$

Free body diagram of body of mass m_2



Writing the Newton's force balance equation, we have

$$m_2 \ddot{y} = -k_1(y - x) - c(\dot{y} - \dot{x}),$$

$$\Rightarrow m_2 \ddot{y} + k_1(y - x) + c(\dot{y} - \dot{x}) = 0. \quad (26)$$

Equations (25) and (26) represent the equations of motion for the quarter car model.

State-space representation

Let the states of the system be defined as

$$\begin{aligned} x &= x_1, \\ \dot{x} &= x_2, \\ y &= x_3, \\ \dot{y} &= x_4. \end{aligned} \quad (27)$$

The following two equations can be derived based on the above relations.

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_3 &= x_4. \end{aligned} \quad (28)$$

Substituting the relations given by equation (27) in equation (25), we get

$$m_1 \ddot{x} + (k_1 + k_2)x - k_1 y - c(\dot{y} - \dot{x}) = k_2 r,$$

$$\Rightarrow m_1 \dot{x}_2 + (k_1 + k_2)x_1 - k_1 x_3 - c(x_4 - x_2) = k_2 r,$$

$$\Rightarrow \dot{x}_2 = \frac{k_2}{m_1} r - \frac{(k_1 + k_2)}{m_1} x_1 - \frac{c}{m_1} x_2 + \frac{k_1}{m_1} x_3 + \frac{c}{m_1} x_4. \quad (29)$$

Similarly substituting the relations given by equation (27) in equation (26), we get

$$m_2 \ddot{y} + k_1(y - x) + c(\dot{y} - \dot{x}) = 0,$$

$$\Rightarrow m_2 \dot{x}_4 + k_1(x_3 - x_1) + c(x_4 - x_2) = 0,$$

$$\Rightarrow \dot{x}_4 = \frac{k_1}{m_2} x_1 + \frac{c}{m_2} x_2 - \frac{k_1}{m_2} x_3 - \frac{c}{m_2} x_4. \quad (30)$$

Rewriting equations (28), (29) and (30) in matrix format, we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{(k_1+k_2)}{m_1} & -\dfrac{c}{m_1} & \dfrac{k_1}{m_1} & \dfrac{c}{m_1} \\ 0 & 0 & 0 & 1 \\ \dfrac{k_1}{m_2} & \dfrac{c}{m_2} & -\dfrac{k_1}{m_2} & -\dfrac{c}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{k_2}{m_1} \\ 0 \\ 0 \end{bmatrix} r. \tag{31}$$

If the output of the system is the displacement of mass m_2, then the output equation can be expressed in the matrix format as

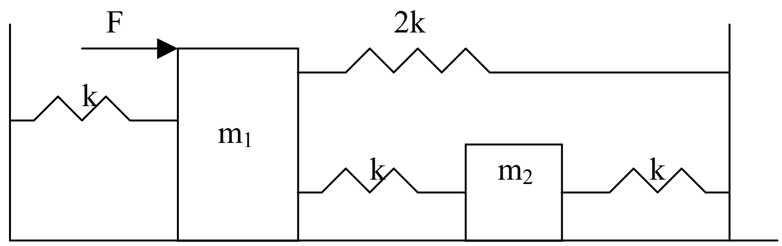
$$Y = y = x_3,$$

$$Y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}. \tag{32}$$

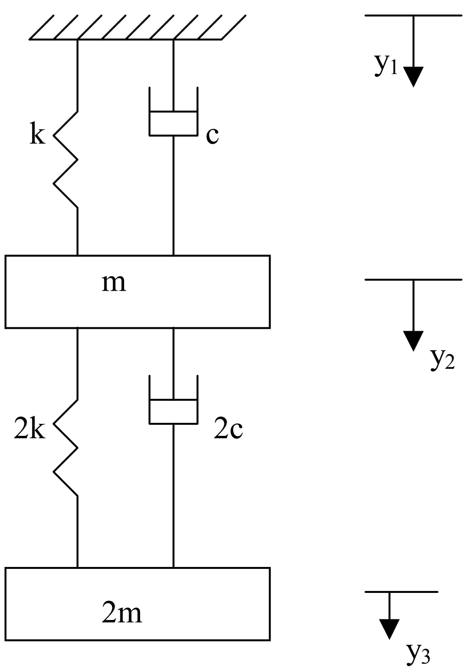
Equations (31) and (32) represent the state-space form of the system defined.

Assignment

1) Derive the differential equation of motion for the system shown below.



2) For the system shown below, derive the governing differential equation of motion.



Recommended Reading

"Feedback Control of Dynamic Systems" 4th Edition, by Gene F. Franklin et.al – pp 24 - 45.

Recommended Assignment

"Feedback Control of Dynamic Systems" 4th Edition, by Gene F. Franklin et.al – problems 2.1, 2.8.