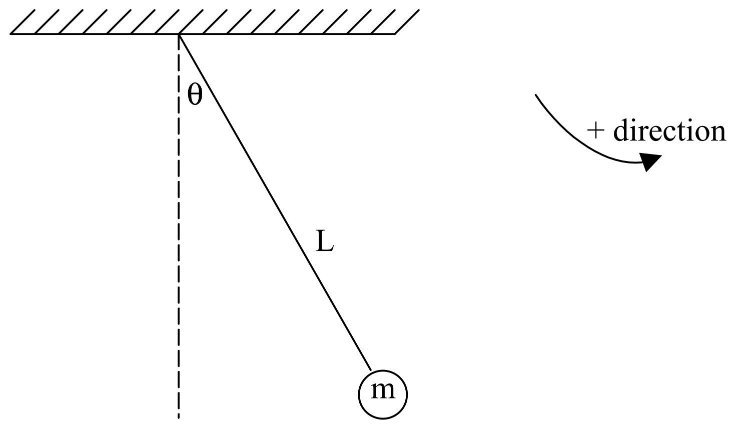
HANDOUT E.6 - EXAMPLES ON MODELLING OF ROTATIONAL MECHANICAL SYSTEMS

Note that the time dependence of variables is ignored for all manipulations.

Example 1: A single DOF system

Consider a simple pendulum shown below.

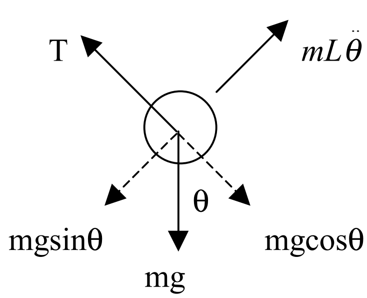


Kinematics stage

There is only one rigid body. Let the degree of freedom of the rigid body of mass, 'm' be defined by the angle θ, moved by the body from the vertical position. Therefore the angular velocity and angular acceleration of the body is given by $\dot{\theta}$ and $\ddot{\theta}$ respectively. Therefore the linear velocity and linear acceleration of the body is given by $L\dot{\theta}$ and $L\ddot{\theta}$ respectively.

Kinetics stage

Free body diagram of the body



T is the tension in the string. Writing the Newton's force balance equation, we get

$$T - mg\cos\theta = 0,$$

$$mL\ddot{\theta} = -mg\sin\theta ,$$

Therefore the governing differential equation of motion for the system is given by

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0 . \qquad (1)$$

For small angles, i.e., if θ is very small then

$$\sin\theta \approx \theta$$

Therefore the equation of motion reduces to

$$\ddot{\theta} + \frac{g}{L}\theta = 0. \qquad (2)$$

Equation (2) represents the final linearized differential equation of motion for a simple pendulum.

The generalized second order differential equation is given by

$$\ddot{\theta} + \omega_n^2\theta = 0 ,$$

where ω_n is the natural frequency of the system.

Comparing equation (2), with the above-generalized equation, we get

$$\omega_n = \sqrt{\frac{g}{L}}.$$

State-space representation

Let the states of the system be defined by

$$\theta = x_1, \qquad \dot{\theta} = x_2. \qquad (3)$$

From the above relations, we get

$$\dot{x}_1 = x_2 . \qquad (4)$$

Substituting the relations given by equation (3) in equation (2), we get

$$\ddot{\theta} + \frac{g}{L}\theta = 0,$$
$$\Rightarrow \dot{x}_2 = -\frac{g}{L}x_1. \qquad (5)$$

Rewriting equations (4) and (5) in matrix format, we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \qquad (6)$$

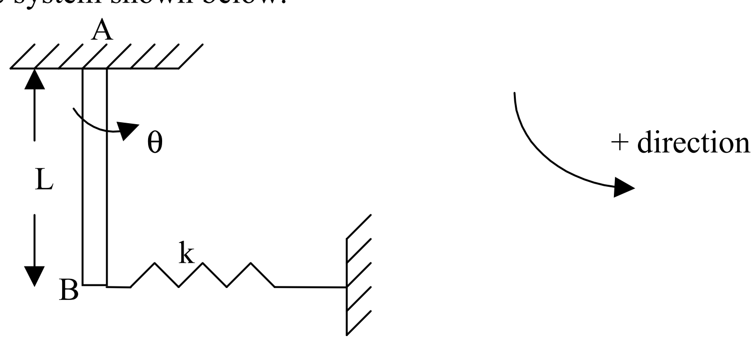
If the output of the system is the angular velocity of the bob of the pendulum, then expressing the output equation in a matrix format, we have

$$y = \dot{\theta} = x_2,$$
$$\Rightarrow y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \qquad (7)$$

Equations (6) and (7) represent the state-space form of the above system.

Example 2: A single DOF system

Consider the system shown below.

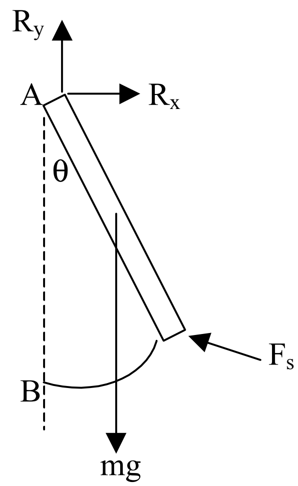


Kinematics stage

From the above figure, it can be seen that there is only one rigid body and the degree of freedom is represented by θ in the counter clockwise direction. The angular velocity and the angular acceleration of the rod are given by $\dot{\theta}$ and $\ddot{\theta}$ respectively. This completes the kinematics stage.

Kinetics stage

Free body diagram of the rod



If we assume that θ is very small, then point B essentially moves horizontally. Therefore the distance moved by point B is the arc length Lθ. This is the amount by which the spring is compressed. Therefore the spring force is given by $F_s = kL\theta$. Furthermore, since θ is small, the spring is essentially horizontal. Therefore the moment of the spring force about the pivot A is given by $-L(kL\theta)$.

Taking moments about the pivot A, we have

$$-kL^2\theta - mg\frac{L}{2}\sin\theta = I_A\ddot{\theta} \tag{8}$$

For small values of θ, we have

$$\sin\theta \approx \theta$$

Therefore equation (8) reduces to

$$I_A\ddot{\theta} + kL^2\theta + mg\frac{L}{2}\theta = 0 \tag{9}$$

Equation (9) represents the governing equation of motion.

State-space representation

Let the states of the system be defined as

$$\theta = x_1,$$

$$\dot{\theta} = x_2. \tag{10}$$

From the above relations, it can be seen that

$$\dot{x}_1 = x_2. \tag{11}$$

Substituting the relations given by equation (10) in equation (9), we have

$$I_A \ddot{\theta} + kL^2\theta + mg\frac{L}{2}\theta = 0,$$

$$\Rightarrow I_A \dot{x}_2 + (kL^2 + \frac{mgL}{2})x_1 = 0,$$

$$\Rightarrow \dot{x}_2 = -\frac{(kL^2 + \frac{mgL}{2})}{I_A}x_1. \tag{12}$$

Rewriting the equations (11) and 12) in matrix format, we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{(kL^2 + \frac{mgL}{2})}{I_A} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{13}$$

If the output of the system is the angular displacement of the rod, then expressing the output relation in matrix format, we have

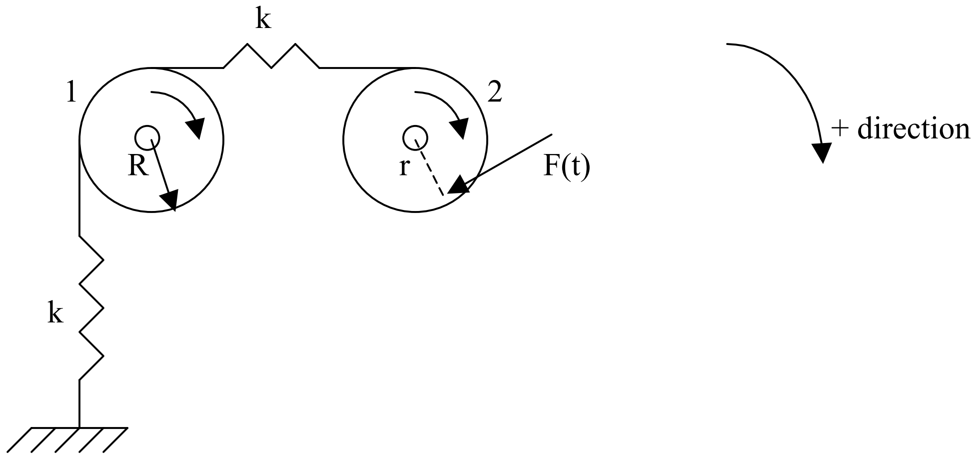
$$y = \theta = x_1$$

$$\Rightarrow y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{14}$$

Equations (13) and (14) represent the state-space form of the above-defined system.

Example 3: Two DOF system

Consider the system shown in the figure below.



Kinematics stage

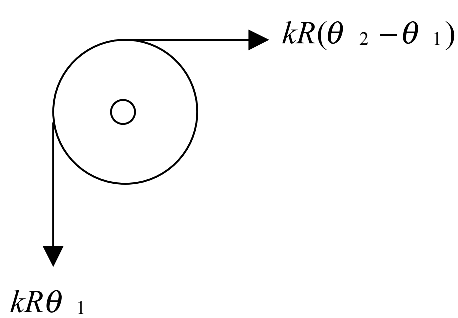
From the above figure it can be seen that, there are two rigid bodies, which are hinged and can only rotate about the fixed point. Since the two bodies are interconnected by a spring, there are two degrees of freedom for the system. Let the degrees of freedom be the rotation of the two discs in the clockwise direction. Let the angle of rotation of the two discs be defined as 'θ_1' and 'θ_2' respectively. Then the angular velocities of the discs are $\dot{\theta}_1$ and $\dot{\theta}_2$ respectively and the angular acceleration of the two discs are given by $\ddot{\theta}_1$ and $\ddot{\theta}_2$ respectively. This completes the kinematics stage.

Kinetics stage

Note that since the discs are identical, they have the same the moment of inertia, 'I'.

Assume θ_2 to be greater than θ_1.

Free body diagram of the disc 1



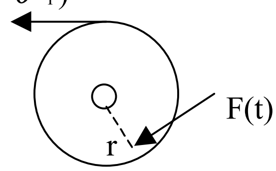
Writing the torque balance equation, i.e., taking moments about the center of the disc, we get

$$\sum M = I\ddot{\theta}$$

$$\Rightarrow -(kR\theta_1)R + kR(\theta_2 - \theta_1)R = I\ddot{\theta}_1,$$

$$\Rightarrow I\ddot{\theta}_1 + 2kR^2\theta_1 - kR^2\theta_2 = 0. \quad (15)$$

Free body diagram of the disc 2



Taking moments about the center of the disc, we have

$$I\ddot{\theta}_2 = -kR(\theta_2 - \theta_1)R + Fr,$$

$$I\ddot{\theta}_2 - kR^2\theta_1 + kR^2\theta_2 = Fr. \quad (16)$$

Equations (15) and (16) represent the equations of motion of the system. Rewriting the equations in matrix format, we have

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 2kR^2 & -kR^2 \\ -kR^2 & kR^2 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ Fr \end{bmatrix}.$$

State-space representation

Let the states of the system be defined as

$$\begin{aligned} \theta_1 &= x_1, \\ \dot{\theta}_1 &= x_2, \\ \theta_2 &= x_3, \\ \dot{\theta}_2 &= x_4. \end{aligned} \quad (17)$$

From the above relations, the following two differential equations can be derived.

$$\dot{x}_1 = x_2, \qquad \dot{x}_3 = x_4. \tag{18}$$

Substituting the relations given by equation (17) in equation (15), we get

$$I\ddot{\theta}_1 + 2kR^2\theta_1 - kR^2\theta_2 = 0,$$
$$\Rightarrow I\dot{x}_2 + 2kR^2 x_1 - kR^2 x_3 = 0,$$
$$\Rightarrow \dot{x}_2 = -\frac{2kR^2}{I}x_1 + \frac{kR^2}{I}x_3. \tag{19}$$

Substituting the relations given by equation (17) in equation (16), we have

$$I\ddot{\theta}_2 - kR^2\theta_1 + kR^2\theta_2 = Fr,$$
$$\Rightarrow I\dot{x}_4 - kR^2 x_1 + kR^2 x_3 = Fr,$$
$$\Rightarrow \dot{x}_4 = \frac{kR^2}{I}x_1 - \frac{kR^2}{I}x_3 + \frac{r}{I}F. \tag{20}$$

Rewriting equations (18), (19) and (20) in matrix format, we get

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{2kR^2}{I} & 0 & \frac{kR^2}{I} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{kR^2}{I} & 0 & -\frac{kR^2}{I} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{r}{I} \end{bmatrix} F. \tag{21}$$

If the output of the system is the angular displacements of both the discs, then expressing the output equation in matrix format, we get

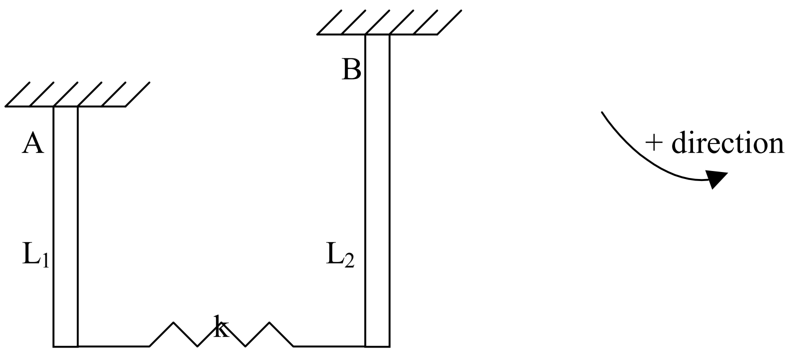
$$y_1 = \theta_1 = x_1,$$
$$y_2 = \theta_2 = x_3,$$

$$\Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}. \tag{22}$$

Equations (21) and (22) represent the state-space form of the system defined above.

Example 4: Two DOF system

Consider the system shown below.



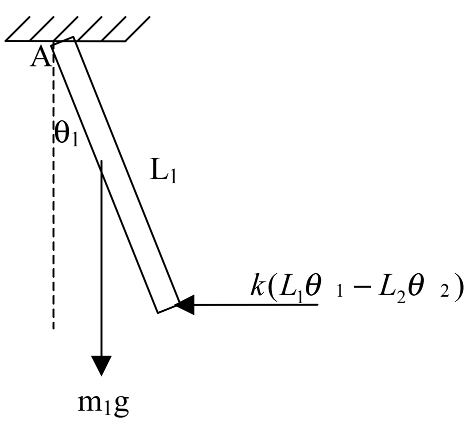
Kinematics stage

There are two rigid bodies and since they are attached by means of a spring, the number of degrees of freedom of the system is two. Let the two DOF's be represented by the amount of angular displacement of both the rods. Therefore let the DOF's be θ_1 and θ_2.

Then the angular velocities and angular accelerations will be $\dot{\theta}_1, \dot{\theta}_2, \ddot{\theta}_1, \ddot{\theta}_2$ respectively. This completes the kinematics stage. Assume θ_1 and θ_2 to be small.

Kinetics stage

Free body diagram of the rod 1



Taking moments about the point A, we have

$$\sum M_A = I_A \ddot{\theta}_1$$
$$\Rightarrow -m_1 g \frac{L_1}{2}\sin\theta_1 - k(L_1\theta_1 - L_2\theta_2)L_1\cos\theta_1 = I_{1A}\ddot{\theta}_1 . \tag{23}$$

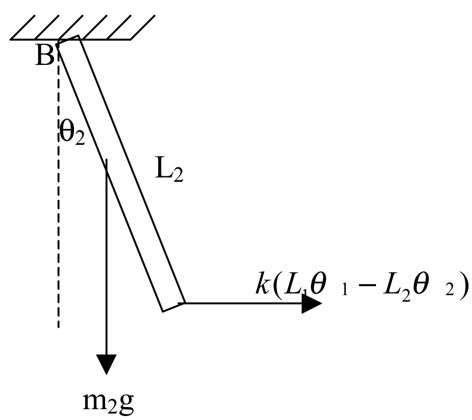
For very small angles,

$$\sin\theta_1 \approx \theta_1,$$
$$\cos\theta_1 \approx 1.$$

Substituting the above relation in equation (23), we have

$$I_{1A}\ddot{\theta}_1 + (kL_1^2 + \frac{m_1 g L_1}{2})\theta_1 - kL_1L_2\theta_2 = 0. \quad (24)$$

Free body diagram of the rod 2



Taking moments about the point B, we get

$$\sum M_B = I_B\ddot{\theta}_2$$
$$\Rightarrow -m_2 g\frac{L_2}{2}\sin\theta_2 + k(L_1\theta_1 - L_2\theta_2)L_2\cos\theta_2 = I_{2B}\ddot{\theta}_2. \quad (25)$$

Assuming small angles, the above equation reduces to

$$I_{2B}\ddot{\theta}_2 + (kL_2^2 + \frac{m_2 g L_2}{2})\theta_2 - kL_1L_2\theta_1 = 0. \quad (26)$$

Equations (24) and (26) represent the governing differential equations of motion.

State-space representation

Let the states of the system be defined as

$$\theta_1 = x_1,$$
$$\dot{\theta}_1 = x_2, \quad (27)$$

$$\theta_2 = x_3, \qquad \dot{\theta}_2 = x_4. \tag{28}$$

From the relations given by equations (27) and (28), we have

$$\dot{x}_1 = x_2, \qquad \dot{x}_3 = x_4. \tag{29}$$

Substituting the relations given by equations (27) and (28) in equation (24), we have

$$I_{1A}\ddot{\theta}_1 + (kL_1^2 + \frac{m_1 g L_1}{2})\theta_1 - kL_1L_2\theta_2 = 0.,$$

$$\Rightarrow I_{1A}\dot{x}_2 + (kL_1^2 + \frac{m_1 g L_1}{2})x_1 - kL_1L_2x_3 = 0,$$

$$\dot{x}_2 = -\frac{(kL_1^2 + \frac{m_1 g L_1}{2})}{I_{1A}}x_1 + \frac{kL_1L_2}{I_{1A}}x_3. \tag{30}$$

Similarly, substituting the relations given by equations (27) and (28) in equation (26), we get

$$I_{2B}\ddot{\theta}_2 + (kL_2^2 + \frac{m_2 g L_2}{2})\theta_2 - kL_1L_2\theta_1 = 0.,$$

$$\Rightarrow I_{2B}\dot{x}_4 + (kL_2^2 + \frac{m_2 g L_2}{2})x_3 - kL_1L_2x_1 = 0,$$

$$\dot{x}_4 = \frac{kL_1L_2}{I_{2B}}x_1 - \frac{(kL_2^2 + \frac{m_2 g L_2}{2})}{I_{2B}}x_3. \tag{31}$$

Rewriting equations (29), (30) and (31) in matrix format, we have

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{(kL_1^2 + \frac{m_1 g L_1}{2})}{I_{1A}} & 0 & \frac{kL_1L_2}{I_{1A}} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{kL_1L_2}{I_{2B}} & 0 & \frac{(kL_2^2 + \frac{m_2 g L_2}{2})}{I_{2B}} & 0 \end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}. \tag{32}$$

If the output if the system is the angular displacement of rod 1, then the output expression can be expressed in matrix format as

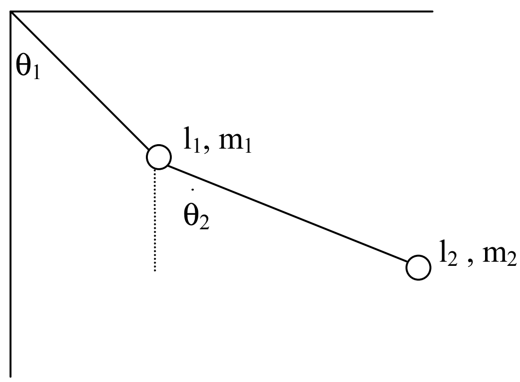
$$y = \theta_1 = x_1,$$

$$\Rightarrow y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}. \qquad (33)$$

Equations (32) and (33) represents the state-space form of the system defined.

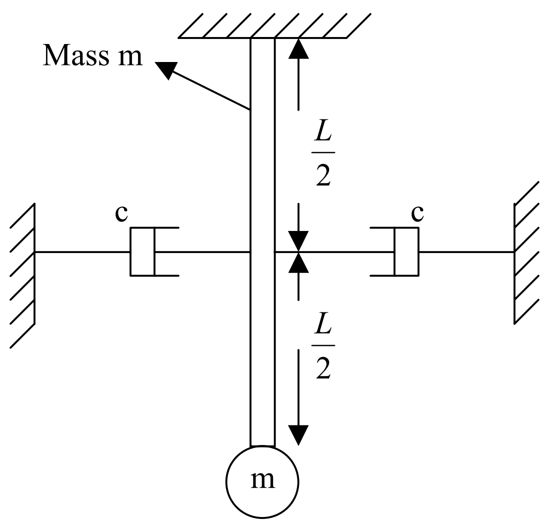
Assignment

1) For the system shown below, derive the governing differential equation.



The above system consists of two point masses m_1 and m_2, each suspended by strings of length L_1 and L_2.

2) Derive the governing differential equations of motion for the system shown below.



Recommended Reading

"Feedback Control of Dynamic Systems" 4th Edition, by Gene F. Franklin et.al – pp 24 - 45.

Recommended Assignment

"Feedback Control of Dynamic Systems" 4th Edition, by Gene F. Franklin et.al – problems 2.3.