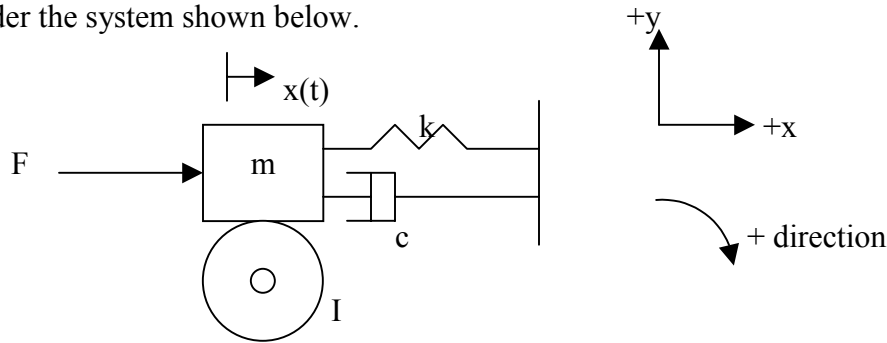# HANDOUT E.7 - EXAMPLES ON MODELLING OF MECHANICAL SYSTEMS – MIXED ROTATIONAL AND TRANSLATIONAL

Note that the time dependence of variables is ignored for all manipulations.

Example 1: One DOF system

Consider the system shown below.

The disc is of radius 'R' and has a moment of inertia 'I'. There is friction between the disc and the block of mass 'm'.

Kinematics stage

From the above figure it can be seen that there are two rigid bodies. The coordinates representing their independent movement is given by 'x' and 'θ' respectively. But since there exists a relation between these two coordinates, which is given by $x = R\theta$, the number of degrees of freedom of the system is one. Let the degree of freedom be 'x'.

The velocity and the acceleration are given by $x,\ \dot{x},\ \theta,\ \dot{\theta}$ respectively. This completes the kinematics stage.
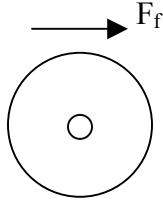
Kinetics stage

Free body diagram of block
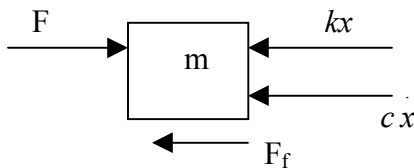
Note that the gravity force is not considered, as the reaction force of the disc on the block balances this force. Writing the Newton's second law of motion, we get

$$\sum F = ma$$

$$\Rightarrow F - kx - c\dot{x} - F_f = m\ddot{x},$$

$$\Rightarrow m\ddot{x} + c\dot{x} + kx + F_f = F. \tag{1}$$

Free body diagram of the disc

Note that the gravity force is ignored as the reaction force due to the block on the disc balances the gravity force.

Taking moments about the center of the disc, we have

$$\sum M = I\ddot{\theta},$$

$$\Rightarrow F_f R = I\ddot{\theta},$$

$$\Rightarrow F_f = \frac{I}{R}\ddot{\theta}.$$

Since $x = R\theta$, we get

$$F_f = \frac{I}{R^2}\ddot{x}.$$

Substituting the value of F_f in equation (1), we get

$$m\ddot{x} + c\dot{x} + kx + F_f = F,$$

$$\Rightarrow m\ddot{x} + c\dot{x} + kx + \frac{I}{R^2}\ddot{x} = F,$$

$$\Rightarrow (m + \frac{I}{R^2})\ddot{x} + c\dot{x} + kx = F. \tag{2}$$

Equation (2) represents the governing equation of motion for the system defined.

State-space representation

Let the states of the system be

$$x = x_1,$$
$$\dot{x} = x_2. \tag{3}$$

From the above relations, we get

$$\dot{x}_1 = x_2. \tag{4}$$

Substituting the relations given by equation (3) in equation (2), we have

$$(m + \frac{I}{R^2})\ddot{x} + c\dot{x} + kx = F,$$

$$\Rightarrow (m + \frac{I}{R^2})\dot{x}_2 + cx_2 + kx_1 = F,$$

$$\dot{x}_2 = \frac{F}{(m + \frac{I}{R^2})} - \frac{k}{(m + \frac{I}{R^2})}x_1 - \frac{c}{(m + \frac{I}{R^2})}x_2. \tag{5}$$

Rewriting the equations (4) and (5) in matrix format, we get

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{(m + \frac{I}{R^2})} & -\frac{c}{(m + \frac{I}{R^2})} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{(m + \frac{I}{R^2})} \end{bmatrix}F. \tag{6}$$

If the output of the system is the displacement of the block, then the output relation can be expressed in matrix format as follows.

$$y = x = x_1,$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{7}$$

Equations (6) and (7) represent the state-space form of the system defined.

Example 2: One DOF system
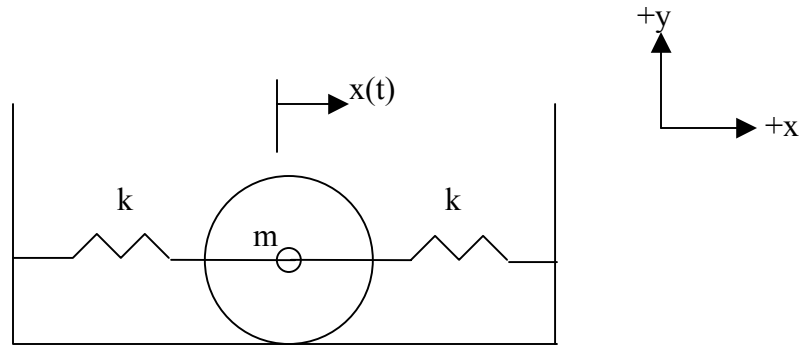
Consider the system shown above. (Assume no friction)

Kinematics stage

There is one rigid body. The number of degrees of freedom of the system is one. Let the degree of freedom be represented as 'x'. The relation between the rotation of the disc and the linear displacement moved by the disc is given by $x = R\theta$ (Assume rolling without slipping). The velocity and the acceleration of the disc are $\dot{x},\ \ddot{x}$ respectively.

Kinetics stage

Free body diagram of the disc
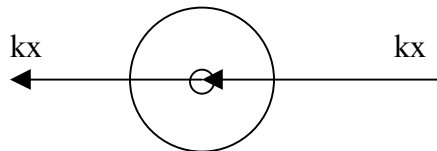
Writing the Newton's second law of motion, we get

$$\sum F = ma$$

$$\Rightarrow -2kx = m\ddot{x},$$

$$m\ddot{x} + 2kx = 0. \tag{8}$$

Since,

$$x = R\theta,$$

Substituting the above relation in equation (8), we get

$$mR\ddot{\theta} + 2kR\theta = 0. \tag{9}$$

Equation (9) represents the governing differential equation of motion.

State-space representation

Let the states of the system be defined as

$$\theta = x_1,$$
$$\dot{\theta} = x_2. \tag{10}$$

From the above relations, the following equation can be written.

$$\dot{x}_1 = x_2. \tag{11}$$

Substituting the relations given by equation (10) in equation (9), we have

$$mR\ddot{\theta} + 2kR\theta = 0,$$

$$\Rightarrow m\dot{x}_2 + 2kx_1 = 0,$$

$$\Rightarrow \dot{x}_2 = -\frac{2k}{m}x_1. \tag{12}$$

Rewriting equations (11) and (12) in matrix format, we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{2k}{m} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{13}$$

If the output of the system is the linear displacement of the disc, then the output relation can be expressed in the matrix format as

$$y = x = R\theta = Rx_1,$$

$$y = \begin{bmatrix} R & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{14}$$

Equations (13) and (14) represent the state-space form of the system defined.

Example 3: Two DOF system
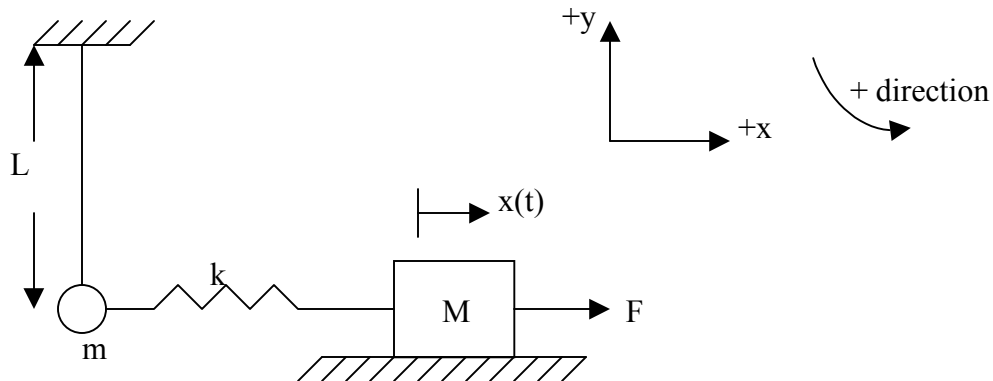
Kinematics stage

There are two rigid bodies and the number of degrees of freedom of the system is two. Let the two degrees of freedom be represented by the angular displacement of the bob, 'θ' and the linear displacement of the block, 'x'. Therefore the linear velocity and the linear acceleration of the block are $\dot{x},\ \ddot{x}$ respectively. Similarly the angular velocity and the angular acceleration of the bob are $\dot{\theta},\ \ddot{\theta}$ respectively. Assume $x > \theta$.

Kinetics stage

Free body diagram of the bob
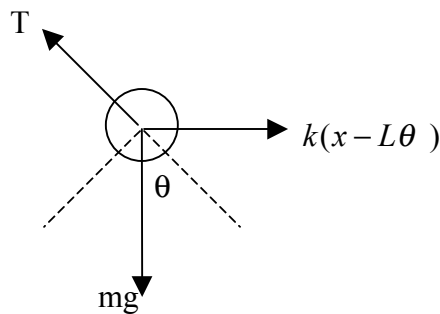
The linear acceleration of the bob is towards the right and is equal to $L\ddot{\theta}$.

Therefore writing the Newton's law of motion, we get

$$\sum F_x = ma,$$

$$\Rightarrow k(x - L\theta) - T\sin\theta = mL\ddot{\theta}.$$

(15)

Similarly

$$\sum F_y = ma$$
$$\Rightarrow T\cos\theta = mg.$$

Substituting the value of tension, T in equation (15), we have

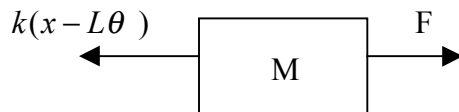$$k(x - L\theta) - mg\tan\theta = mL\ddot{\theta}. \tag{16}$$

For small angles, we have

$$\tan\theta \approx \theta.$$

Therefore equation (16) reduces to

$$k(x - L\theta) - mg\theta = mL\ddot{\theta}. \tag{17}$$

Free body diagram of block

$$k(x - L\theta) \quad \overleftarrow{\qquad \boxed{\begin{array}{c} \\ M \\ \ \end{array}} \qquad} \quad F \longrightarrow$$

Note that the gravity force is neglected as the normal reaction force of the ground balances the gravity force. Writing the Force balance equation, we get

$$\sum F = ma$$
$$\Rightarrow F - k(x - L\theta) = M\ddot{x},$$
$$\Rightarrow M\ddot{x} + k(x - L\theta) = F. \tag{18}$$

Equations (17) and (18) represent the governing differential equation.

State-space representation

Let the states of the system be defined as

$$\theta = x_1,$$
$$\dot{\theta} = x_2,$$
$$x = x_3, \tag{19}$$
$$\dot{x} = x_4.$$

From the above relations, the following two equations can be derived.

$$\dot{x}_1 = x_2,$$ (20)

$$\dot{x}_3 = x_4.$$

Substituting the relations given by equation (19) in equation (17), we get

$$k(x - L\theta) - mg\theta = mL\ddot{\theta},$$

$$\Rightarrow k(x_3 - Lx_1) - mgx_1 = mL\dot{x}_2,$$

$$\Rightarrow \dot{x}_2 = -\frac{(kL + mg)}{mL}x_1 + \frac{k}{mL}x_3.$$ (21)

Similarly substituting the relations given by equation (19) in equation (18), we get

$$M\ddot{x} + k(x - L\theta) = F,$$

$$\Rightarrow M\dot{x}_4 + k(x_3 - Lx_1) = F,$$

$$\Rightarrow \dot{x}_4 = \frac{F}{M} + \frac{kL}{M}x_1 - \frac{k}{M}x_3.$$ (22)

Rewriting equations (20), (21) and (22) in matrix format, we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{(kL + mg)}{mL} & 0 & \dfrac{k}{mL} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{kL}{M} & 0 & -\dfrac{k}{M} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{1}{M} \end{bmatrix} F.$$ (23)
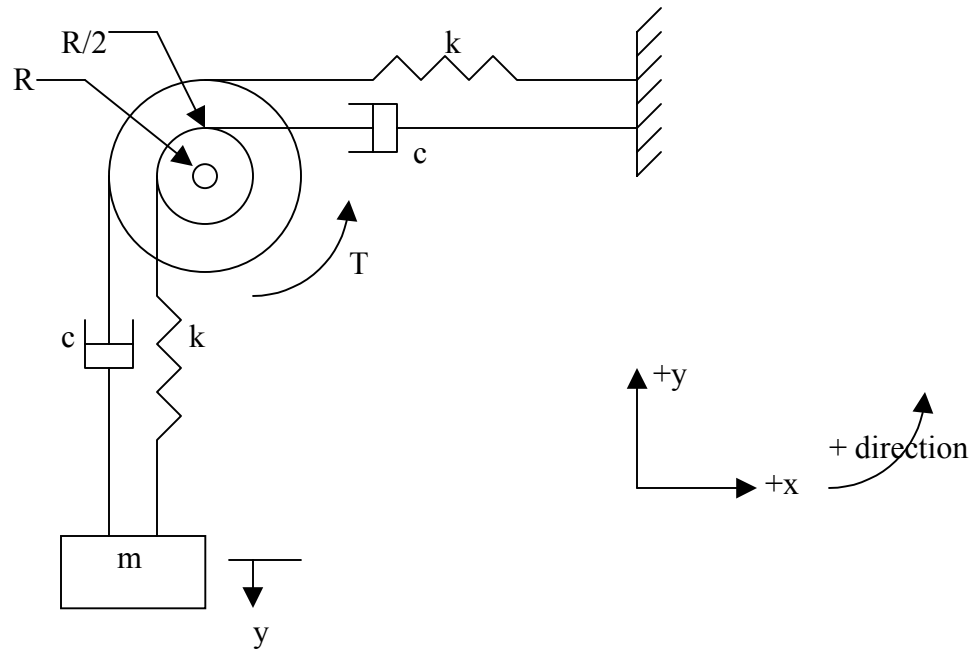
If the output of the system is the linear displacement of the block, then the output relation can be expressed in the matrix format as

$$y = x = x_3,$$

$$y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$ (24)

Equations (23) and (24) represent the state-space form of the above-defined system.

Example 4: Two DOF system

Consider the system shown below.

Moment of inertia of the drum is equal to I.

Kinematics stage

There are two rigid bodies and they can move independent of each other. Therefore the number of degrees of freedom of the system is two. Let the degrees of freedom be chosen as the linear displacement of the block, 'y' and the angular displacement of the drum, 'θ'.

Therefore the linear velocity and the linear acceleration of the block is given by $\dot{y}$, $\ddot{y}$ respectively. Similarly the angular velocity and the angular acceleration of the drum is given by $\dot{\theta}$, $\ddot{\theta}$ respectively. Assume $y > \theta$.

Kinetics stage

Free body diagram of the block

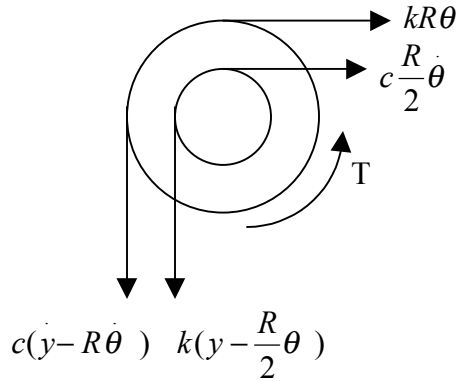$$c(\dot{y} - R\dot{\theta}\ ) \quad k(y - \frac{R}{2}\theta\ )$$
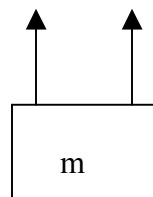
Note that the displacements chosen are from the static equilibrium position. Hence spring force due to the initial elongation of the spring balances the gravity force.

Writing the Newton's second law of motion, we have

$$\sum F_y = ma,$$

$$\Rightarrow c(\dot{y} - R\dot{\theta}) + k(y - \frac{R}{2}\theta) = -m\ddot{y},$$

$$m\ddot{y} + c(\dot{y} - R\dot{\theta}) + k(y - \frac{R}{2}\theta) = 0. \tag{25}$$

Free body diagram of the drum

Taking moments about the center of the drum, we have

$$\sum M = I\ddot{\theta},$$

$$\Rightarrow T + c(\dot{y} - R\dot{\theta})R + k(y - \frac{R}{2}\theta)\frac{R}{2} - kR\theta(R) - c\frac{R}{2}\dot{\theta}(\frac{R}{2}) = I\ddot{\theta},$$

$$\Rightarrow I\ddot{\theta} + (cR^2 + \frac{cR^2}{4})\dot{\theta} - cR\dot{y} + (kR^2 + \frac{kR^2}{4})\theta - k\frac{R}{2}y = T. \tag{26}$$

Equations (25) and (26) represent the governing differential equations of motion.

State-space representation

$$y = x_1,$$

$$\dot{y} = x_2,$$

$$\theta = x_3, \tag{27}$$

$$\dot{\theta} = x_4.$$

From the above relations, the following equations can be deduced.

$$\dot{x}_1 = x_2,$$
$$\dot{x}_3 = x_4.$$

(28)

Substituting the relations given by equation (27) in equation (25), we get

$$m\ddot{y} + c(\dot{y} - R\dot{\theta}) + k(y - \frac{R}{2}\theta) = 0,$$

$$\Rightarrow m\dot{x}_2 + c(x_2 - Rx_4) + k(x_1 - \frac{R}{2}x_3) = 0,$$

$$\Rightarrow \dot{x}_2 = -\frac{k}{m}x_1 - \frac{c}{m}x_2 + \frac{kR}{2m}x_3 + \frac{cR}{m}x_4.$$

(29)

Similarly substituting the relations given by equation (27) in equation (26), we get

$$I\ddot{\theta} + (cR^2 + \frac{cR^2}{4})\dot{\theta} - cR\dot{y} + (kR^2 + \frac{kR^2}{4})\theta - k\frac{R}{2}y = T,$$

$$\Rightarrow I\dot{x}_4 + \frac{5cR^2}{4}x_4 - cRx_2 + \frac{5kR^2}{4}x_3 - k\frac{R}{2}x_1 = T,$$

$$\Rightarrow \dot{x}_4 = \frac{T}{I} + \frac{kR}{2I}x_1 + \frac{cR}{I}x_2 - \frac{5kR^2}{4I}x_3 - \frac{5cR^2}{4I}x_4.$$

(30)

Rewriting equations (28), (29) and (30) in matrix format, we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{k}{m} & -\dfrac{c}{m} & \dfrac{kR}{2m} & \dfrac{cR}{2m} \\ 0 & 0 & 0 & 1 \\ \dfrac{kR}{2I} & \dfrac{cR}{I} & -\dfrac{5kR^2}{4I} & -\dfrac{5cR^2}{4I} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{1}{I} \end{bmatrix} T.$$

(31)

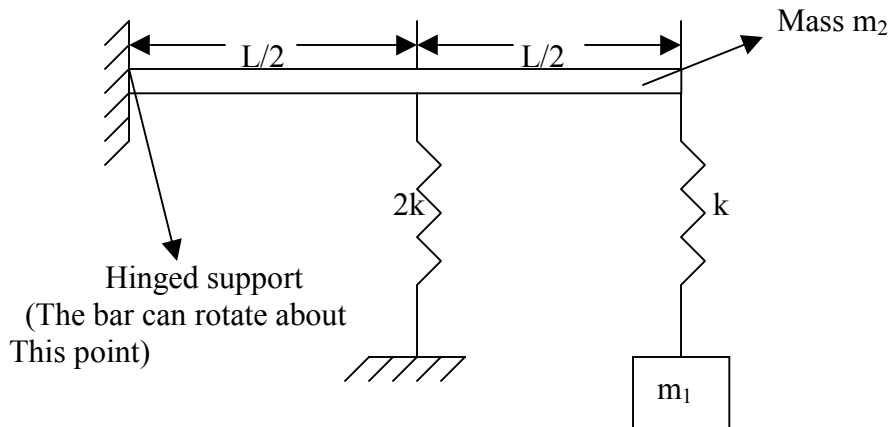If the output of the system is the displacement moved by the block, then the output equation can be expressed in the matrix format as

$$Y = y = x_1,$$

$$Y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$
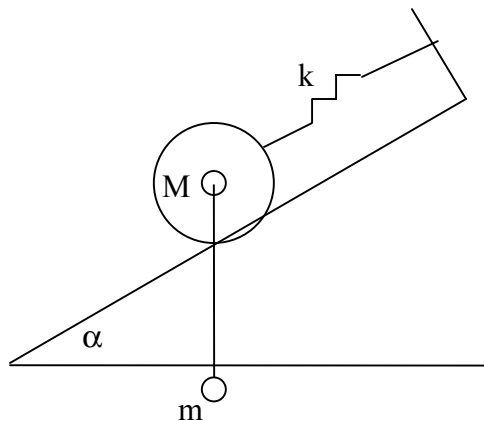
(32)

Equations (31) and (32) represent the state-space form of the system defined.

Assignment

1) Derive the governing differential equation of motion for the following system.

2) Consider the system shown below.

The above figure consists of a drum, to the center of which a simple pendulum is attached. The bob of the pendulum is displaced slightly to the right. Derive the governing differential equation of motion. Assume small angle motion.

Recommended Reading

"Feedback Control of Dynamic Systems" 4th Edition, by Gene F. Franklin et.al – pp 24 - 45.