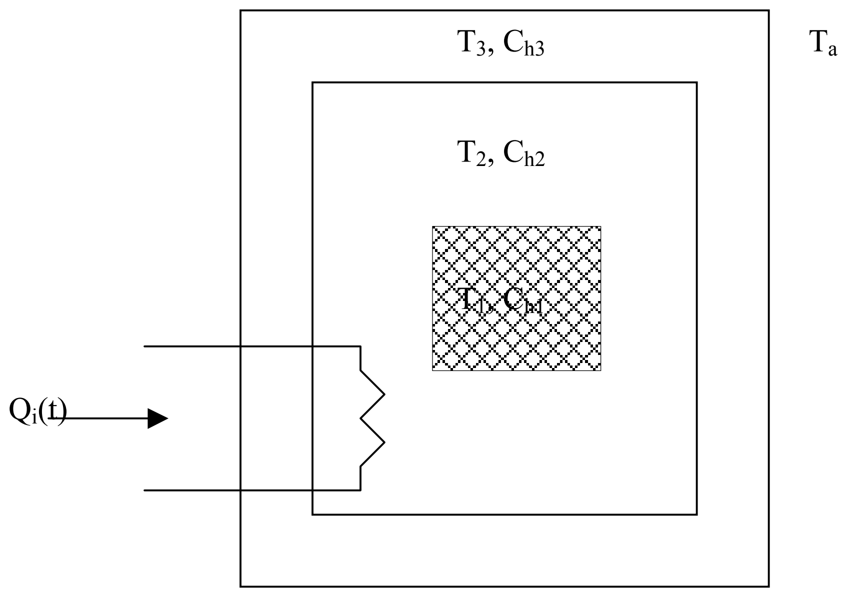
HANDOOUT E.9 - EXAMPLES ON FLUID, THERMAL AND MIXED SYSTEMS

Example 1: A thermal system

The following figure shows a simple model of an industrial furnace. A packing of temperature T_1 is being heated in the furnace by an electric heater supplying heat at the rate $Q_i(t)$. The temperature inside the furnace is T_2, the walls are at temperature T_3 and the ambient temperature is T_a. The thermal capacitances of the packing, the air inside the furnace and the furnace walls are C_{h1}, C_{h2} and C_{h3} respectively. Derive the state-variable equations for this system assuming that the heat is transferred by convection only, with the convective heat transfer coefficients h_{c1} (air-packing), h_{c2} (air-inside walls) and h_{c3} (outside walls-ambient air).



The rate of heat transfer, Q between a solid wall and a fluid flowing over it is given by

$$Q = h_c A(T_w - T_f), \tag{1}$$

where h_c is the convective heat transfer coefficient, A is the area of heat transfer and T_w and T_f represent the wall and fluid temperatures respectively.

Using the above relations for the packing, we have

$$Q_1 = m_1 c_1 \frac{dT_1}{dt} = C_{h1} \frac{dT_1}{dt} = h_{c1} A_1 (T_2 - T_1). \tag{2}$$

Similarly applying the relation for the furnace, we have

$$Q_2 = m_2 c_2 \frac{dT_2}{dt} = C_{h2} \frac{dT_2}{dt} = Q_i(t) - h_{c1} A_1 (T_2 - T_1) - h_{c2} A_2 (T_2 - T_3). \tag{3}$$

Applying the relation given by equation (1) to the walls, we get

$$Q_3 = m_3 c_3 \frac{dT_3}{dt} = C_{h3} \frac{dT_3}{dt} = h_{c2} A_2 (T_2 - T_3) - h_{c3} A_3 (T_3 - T_a). \tag{4}$$

Equations (2), (3) and (4) represent the governing differential equations of motion for the above-defined system.

State-space representation

Let the states of the system be defined as

$$\begin{aligned} T_1 &= x_1, \\ T_2 &= x_2, \\ T_3 &= x_3. \end{aligned} \tag{5}$$

Substituting the above relation in equation (2), we have

$$\begin{aligned} &C_{h1} \frac{dT_1}{dt} = h_{c1} A_1 (T_2 - T_1), \\ &\Rightarrow \dot{x}_1 = -\frac{h_{c1} A_1}{C_{h1}} x_1 + \frac{h_{c1} A_1}{C_{h1}} x_2. \end{aligned} \tag{6}$$

Similarly substituting the relations given by equation (5) in equation (3), we have

$$\begin{aligned} &C_{h2} \frac{dT_2}{dt} = Q_i(t) - h_{c1} A_1 (T_2 - T_1) - h_{c2} A_2 (T_2 - T_3), \\ &\Rightarrow \dot{x}_2 = \frac{h_{c1} A_1}{C_{h2}} x_1 - \left[\frac{h_{c1} A_1}{C_{h2}} + \frac{h_{c2} A_2}{C_{h2}} \right] x_2 + \frac{h_{c2} A_2}{C_{h2}} x_3 + \frac{1}{C_{h2}} Q_i(t). \end{aligned} \tag{7}$$

Substituting the relations given by equation (5) in equation (4), we get

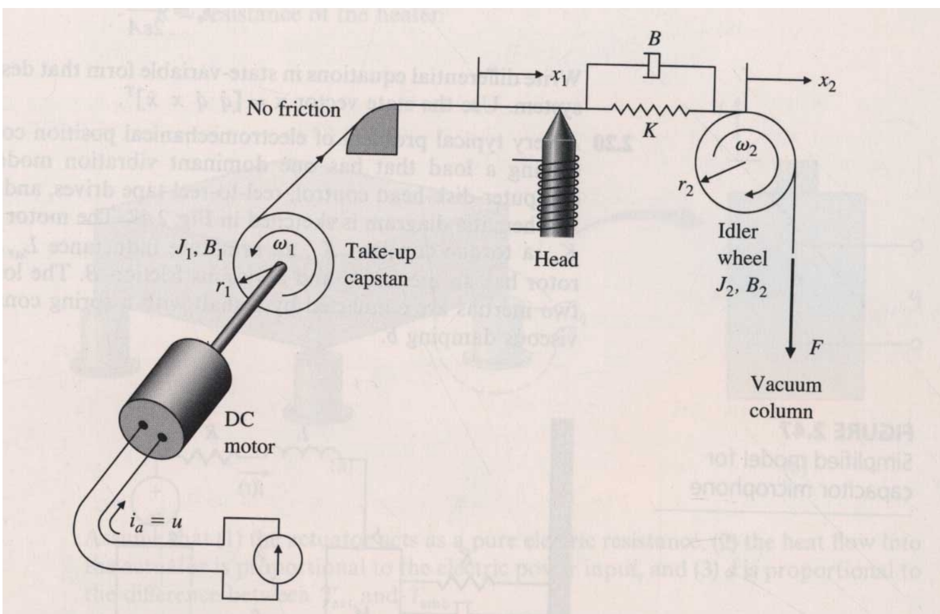
$$\begin{aligned} &C_{h3} \frac{dT_3}{dt} = h_{c2} A_2 (T_2 - T_3) - h_{c3} A_3 (T_3 - T_a), \\ &\Rightarrow \dot{x}_3 = \frac{h_{c2} A_2}{C_{h3}} x_2 - \left[\frac{h_{c2} A_2}{C_{h3}} + \frac{h_{c3} A_3}{C_{h3}} \right] x_3 + \frac{h_{c3} A_3}{C_{h3}} T_a. \end{aligned} \tag{8}$$

Rewriting equations (6), (7) and (8) in matrix format, we have

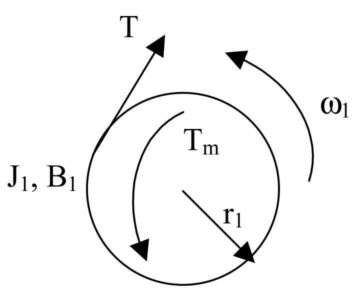
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\dfrac{h_{c1}A_1}{C_{h1}} & \dfrac{h_{c1}A_1}{C_{h1}} & 0 \\ \dfrac{h_{c1}A_1}{C_{h2}} & -\left[\dfrac{h_{c1}A_1}{C_{h2}} + \dfrac{h_{c2}A_2}{C_{h2}}\right] & \dfrac{h_{c2}A_2}{C_{h2}} \\ 0 & \dfrac{h_{c2}A_2}{C_{h3}} & -\left[\dfrac{h_{c2}A_2}{C_{h3}} + \dfrac{h_{c3}A_3}{C_{h3}}\right] \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \dfrac{1}{C_{h2}} & 0 \\ 0 & \dfrac{h_{c3}A_3}{C_{h3}} \end{bmatrix} \begin{bmatrix} Q_i(t) \\ T_a \end{bmatrix}.$$

Example 2: A mixed system

A simplified sketch of a computer tape drive is shown below. Write the equations of motion in terms of the parameters listed below.



Free body diagram of the take-up capstan



where J_1 is the inertial of the motor, B_1 is the motor damping constant, T_m is the torque developed by the motor, T is the tension in the string.

Writing the torque balance equation, we have

$$J_1 \frac{d\omega_1}{dt} + B_1\omega_1 - Tr_1 = T_m. \tag{9}$$

But from the figure it can be concluded that

$$T = B(\dot{x}_2 - \dot{x}_1) + k(x_2 - x_1). \tag{10}$$

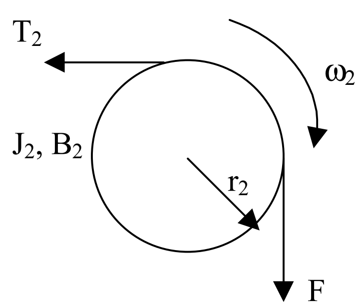
We also know that the torque developed by the motor is proportional to the armature current. Hence

$$T_m = k_t i_a. \tag{11}$$

Substituting the equations (10) and (11) in equation (9), we have

$$\begin{aligned} &J_1 \frac{d\omega_1}{dt} + B_1\omega_1 - Tr_1 - T_m = 0, \\ &\Rightarrow J_1 \frac{d\omega_1}{dt} + B_1\omega_1 - Br_1(\dot{x}_2 - \dot{x}_1) - kr_1(x_2 - x_1) - k_t i_a = 0. \end{aligned} \tag{12}$$

Free body diagram of the idler wheel



Writing the Torque balance equation, we get

$$J_2 \frac{d\omega_2}{dt} + B_2\omega_2 + T_2 r_2 - Fr_2 = 0. \tag{13}$$

But again,

$$T_2 = B(\dot{x}_2 - \dot{x}_1) + k(x_2 - x_1). \tag{14}$$

Substituting equation (14) in equation (13), we get

$$J_2 \frac{d\omega_2}{dt} + B_2\omega_2 + T_2 r_2 - F r_2 = 0,$$
$$\Rightarrow J_2 \frac{d\omega_2}{dt} + B_2\omega_2 + B r_2 (\dot{x}_2 - \dot{x}_1) + k r_2 (x_2 - x_1) - F r_2 = 0. \tag{15}$$

From the figure, the following relation can be concluded,

$$\dot{x}_1 = r_1 \omega_1,$$
$$\dot{x}_2 = r_2 \omega_2. \tag{16}$$

Equations (12), (15) and (16) represent the governing differential equation of motion.

State-space representation

Let the states of the system be defined as

$$x_1 = X_1,$$
$$\omega_1 = X_2,$$
$$x_2 = X_3,$$
$$\omega_2 = X_4. \tag{17}$$

Substituting the above relation in equation (16), we have

$$\dot{x}_1 = r_1 \omega_1,$$
$$\Rightarrow \dot{X}_1 = r_1 X_2. \tag{18}$$
$$\dot{x}_2 = r_2 \omega_2,$$
$$\Rightarrow \dot{X}_3 = r_2 X_4. \tag{19}$$

Substituting the relation given by equation (17) in equation (12), we have

$$J_1 \frac{d\omega_1}{dt} + B_1\omega_1 - B r_1 (\dot{x}_2 - \dot{x}_1) - k r_1 (x_2 - x_1) - k_t i_a = 0,$$
$$\Rightarrow J_1 \dot{X}_2 + B_1 X_2 - B r_1 (r_2 X_4 - r_1 X_2) - k r_1 (X_3 - X_1) - k_t i_a = 0,$$
$$\Rightarrow \dot{X}_2 = -\frac{k r_1}{J_1} X_1 - \frac{\left(B_1 + B r_1^2\right)}{J_1} X_2 + \frac{k r_1}{J_1} X_3 + \frac{B r_1 r_2}{J_1} X_4 + \frac{k_t}{J_1} i_a. \tag{20}$$

Substituting the relation given by equation (17) in equation (15), we have

$$J_2 \frac{d\omega_2}{dt} + B_2\omega_2 + Br_2(\dot{x}_2 - \dot{x}_1) + kr_2(x_2 - x_1) - Fr_2 = 0,$$

$$\Rightarrow J_2 \dot{X}_4 + B_2 X_4 + Br_2(r_2 X_4 - r_1 X_2) + kr_2(X_3 - X_1) - Fr_2 = 0,$$

$$\Rightarrow \dot{X}_4 = \frac{kr_2}{J_2}X_1 + \frac{Br_1r_2}{J_2}X_2 - \frac{kr_2}{J_2}X_3 - \frac{\left(B_2 + Br_2^2\right)}{J_2}X_4 + \frac{r_2}{J_2}F. \tag{21}$$

Rewriting equations (18), (19), (20) and (21) in matrix format, we have

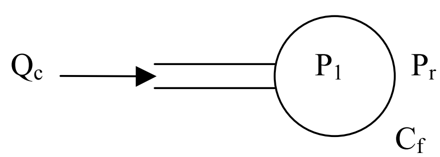
$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \\ \dot{X}_4 \end{bmatrix} = \begin{bmatrix} 0 & r_1 & 0 & 0 \\ -\frac{kr_1}{J_1} & -\frac{\left(B_1 + Br_1^2\right)}{J_1} & \frac{kr_1}{J_1} & \frac{Br_1r_2}{J_1} \\ 0 & 0 & 0 & r_2 \\ \frac{kr_2}{J_2} & \frac{Br_1r_2}{J_2} & -\frac{kr_2}{J_2} & -\frac{\left(B_2 + Br_2^2\right)}{J_2} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{k_t}{J_1} & 0 \\ 0 & 0 \\ 0 & \frac{r_2}{J_2} \end{bmatrix} \begin{bmatrix} i_a \\ F \end{bmatrix}. \tag{22}$$

Example 3: A fluid system

Introduction

Fluid capacitor

A fluid capacitor is shown in the following figure

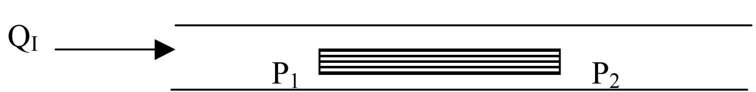


The pressure in a fluid capacitor must be referred to a reference pressure P_r. The volume flow rate Q_c is given by

$Q_c = C_f \dfrac{dP_{1r}}{dt}$, where C_f is the fluid capacitance.

Fluid inertor

The symbolic diagram of a fluid inertor is shown in the following figure.



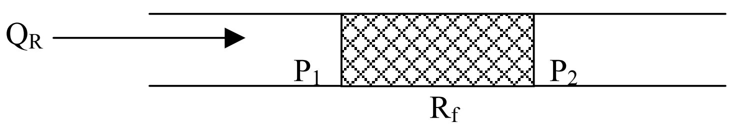
I

The elemental equation for the inertor is

$P_{12} = I \frac{dQ_I}{dt}$, where I is the fluid inertance. For frictionless incompressible flow in a uniform passage having cross sectional area A and length L, the inertance $I = \frac{\rho \ L}{A}$, where ρ is the mass density of the fluid.

Fluid resistor

The symbolic diagram of a fluid resistor is shown below.



The elemental equation of an ideal resistor is

$$P_{12} = R_f Q_R.$$

Problem: Develop the input-output differential equation relating the output pressure to the input pressure for the fluid system shown below.

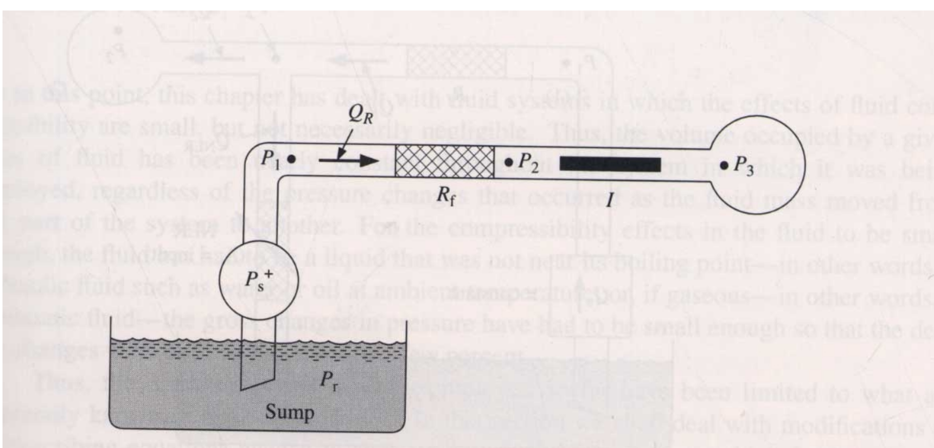


Figure 9.9 A simple fluid *R, I, C* system.

For the fluid resistor, we have

$$P_{12} = R_f Q_R. \quad (23)$$

For the inertor, we get

$$P_{23} = I\frac{dQ_R}{dt}. \tag{24}$$

For the fluid capacitor,

$$Q_R = C_f\frac{dP_{3r}}{dt}. \tag{25}$$

Writing the pressure balance equation, we have

$$P_s = P_{1r} = P_{12} + P_{23} + P_{3r}$$
$$\Rightarrow P_s = R_f Q_R + I\frac{dQ_R}{dt} + P_{3r}$$
$$\Rightarrow P_s = R_f C_f\frac{dP_{3r}}{dt} + C_f I\frac{d^2P_{3r}}{dt^2} + P_{3r},$$
$$\Rightarrow C_f I\frac{d^2P_{3r}}{dt^2} + R_f C_f\frac{dP_{3r}}{dt} + P_{3r} = P_s. \tag{26}$$

Equation (26) represents the governing differential equation of motion for the fluid system shown.

State-space representation

Let the states of the system be defined as

$$P_{3r} = x_1,$$
$$\frac{dP_{3r}}{dt} = x_2. \tag{27}$$

From the above relation, we get

$$\dot{x}_1 = x_2. \tag{28}$$

Substituting the relation given by equation (27) in equation (26), we have

$$C_f I\frac{d^2P_{3r}}{dt^2} + R_f C_f\frac{dP_{3r}}{dt} + P_{3r} = P_s,$$
$$\Rightarrow C_f I\dot{x}_2 + R_f C_f x_2 + x_1 = P_s,$$

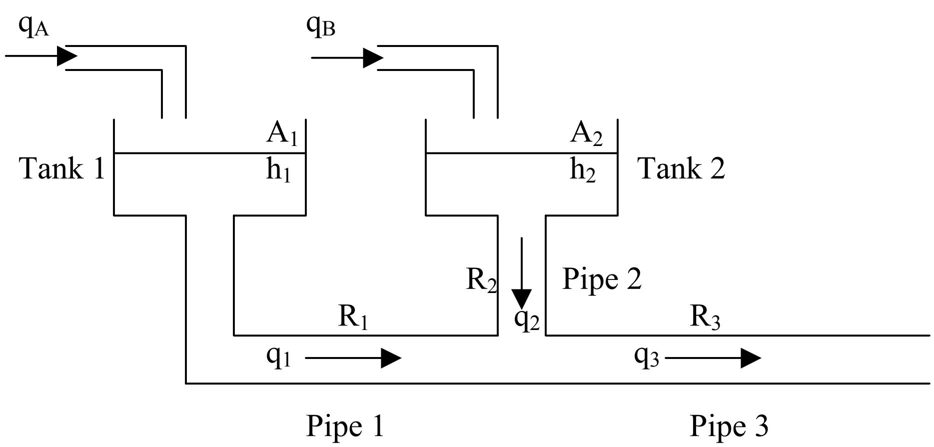
$$\Rightarrow \dot{x}_2 = -\frac{1}{C_f I}x_1 - \frac{R_f}{I}x_2 + \frac{1}{C_f I}P_s. \tag{29}$$

Rewriting equations (28) and (29) in matrix format, we get

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{C_f I} & -\frac{R_f}{I} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{C_f I} \end{bmatrix} P_s. \tag{30}$$

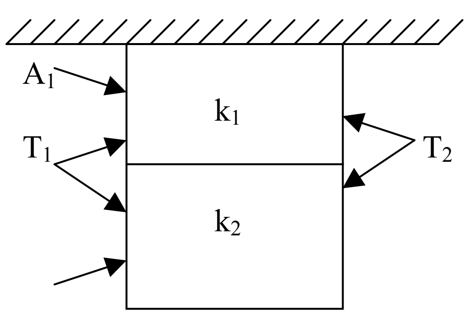
Assignment

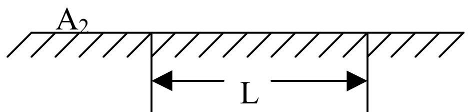
1) The sewage system leading to a treatment plant is shown. The variables q_A and q_B are input flow rates into tanks 1 and 2 respectively. Pipes 1, 2 and 3 have resistances as shown. Derive the state equations.



2) The temperatures of the side surfaces of the composite slab shown below areT_1 and T_2. The other surfaces are perfectly insulated. The cross sectional areas of the two parts of the slab are A_1 and A_2 and their conductivities are k_1 and k_2 respectively. The length of the slab is L.

a) Find the equivalent thermal resistance of the slab and express it in terms of the thermal resistances of the two parts.





3) Write the equations of motion for the hanging crane shown below. Assume that the driving force on the hanging crane is provided by the motor mounted on the cab with one of the support wheels connected directly to the armature shaft. The motor constants are K_e, K_t and the circuit driving the motor has a resistance Ra and no inductance. The wheel has a radius 'r'.