HANDOUT M.3 - EIGEN VALUES AND EIGEN VECTORS

Definition: Let **A** be an n x n matrix. Then a real number λ is called an ***eigenvalue*** of the matrix, **A** if and only if, there is a n-dimensional nonzero vector, **v** for which

$$\mathbf{A}\mathbf{v}_\mathbf{i} = \lambda_i \mathbf{v}_\mathbf{i} \quad ; \; i = 1,\ldots,n \tag{1}$$

Any such vector, **v** is called an ***eigenvector*** of the matrix **A**, associated with the eigenvalue λ.

Example 1: Consider the 2 x 2 matrix given by

$$\mathbf{A} = \begin{bmatrix} 7 & -1 \\ 6 & 2 \end{bmatrix}$$

Then since

Eigenvector

$$\begin{bmatrix} 7 & -1 \\ 6 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 7\times1+(-1)\times3 \\ 6\times1+2\times3 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Eigenvalue

we see that λ = 4 is the eigenvalue of **A**, associated with the eigenvector

$$\mathbf{v}_\mathbf{1} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Also since

$$\begin{bmatrix} 7 & -1 \\ 6 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

We see that λ = 5 is also an eigenvalue of **A**, associated with the eigenvector

$$\mathbf{v}_\mathbf{2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

From the above example, note that the vector $\mathbf{v_1}$ is not the only eigenvector associated with matrix **A**. Any nonzero scalar multiple of vector the $\mathbf{v_1}$ is also an eigenvector of **A** associated with the eigenvalue $\lambda = 4$. In other words

$$\mathbf{v_1^*} = \mathbf{2.v_1} = 2.\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

which is twice the vector $\mathbf{v_1}$ is also an eigenvector of **A**, associated with eigenvalue $\lambda = 4$. A similar statement holds for the eigenvector $\mathbf{v_2}$.

Note:

1. Eigenvalues and eigenvectors are defined only for square matrices i.e. the number of rows must be equal to the number of columns in the matrix.

2. In the above example, the number of rows and number of columns is equal to 2. So the size of the matrix is represented as 2 x 2. The number of eigenvalues associated with the matrix is equal to the size of the matrix i.e. in this case there are two eigenvalues associated with the defined matrix. In general, if the size of the matrix is 'n x n' then there are 'n' eigenvalues associated with the matrix and each one has an associated eigenvector with it.

Procedure for calculating eigenvalues and eigenvectors analytically

If **A** is any square matrix of size n x n and λ is an associated eigenvalue, rewrite equation (1) as,

$$\mathbf{Av_i} = \lambda\ \mathbf{Iv_i}$$

where **I** is the identity matrix of size n x n. (Note: The size of the identity matrix has to be the same as that of the matrix for which the eigenvalues and eigenvectors has to be calculated.)

The above equation reduces to

$$(\mathbf{A} - \lambda\ \mathbf{I})\mathbf{v_i} = 0 \tag{2}$$

which is nothing more than a homogenous system of 'n' equations in 'n' variables.

In order to find the value of λ, the above system of linear equations has to be solved.

$\mathbf{v} = 0$ is a trivial solution, but from the definition of eigenvectors, **v** has to be a nonzero vector.

The eigenvalues of **A** are those real numbers λ for which the homogenous system defined by equation (2) has a non-trivial (or nonzero) solution and the eigenvectors of **A** associated with λ are the nonzero solutions of this system.

Equation (2) has a nonzero solution if and only if its coefficient matrix is noninvertible. The coefficient matrix is noninvertible if and only if its determinant is equal to zero, i.e.,

$$\left|\mathbf{A} - \lambda\,\mathbf{I}\right| = 0. \tag{3}$$

Equation (3) is called the ***characteristic equation*** of matrix **A**. The above procedure is further explained with the help of an example.

Example 2: Let **A**, a 3 x 3 matrix be defined as

$$\mathbf{A} = \begin{bmatrix} 0 & 5 & 7 \\ -2 & 7 & 7 \\ -1 & 1 & 4 \end{bmatrix}$$

The size of **A** is 3 x 3. So the identity matrix **I** has to be

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The characteristic equation is defined as

$$\left|\mathbf{A} - \lambda\,\mathbf{I}\right| = \begin{vmatrix} -\lambda & 5 & 7 \\ -2 & 7-\lambda & 7 \\ -1 & 1 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (\lambda - 2)(\lambda - 4)(\lambda - 5) = 0.$$

The eigenvalues of **A** are the solutions to this equation, namely,

$$\lambda_1 = 2; \;\; \lambda_2 = 4; \;\; \textit{and} \;\; \lambda_3 = 5;$$

To find the eigenvectors, substituting the value of λ = 5 in the coefficient matrix of equation (2), we have

$$(\mathbf{A}-5\mathbf{I})\mathbf{v} = \begin{bmatrix} -5 & 5 & 7 \\ -2 & 2 & 7 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

Rewriting the above matrix in the form of homogenous equations, we get

$$\begin{aligned} -5v_1 + 5v_2 + 7v_3 &= 0 \\ -2v_1 + 2v_2 + 7v_3 &= 0 \\ -v_1 + v_2 - v_3 &= 0 \end{aligned} \tag{4}$$

By multiplying the third equation of equation (4) by (-2) and add the result to the second equation of equation (4), we get

$$v_3 = 0$$

Now by substituting this result in the third equation of equation (4), we get

$$v_1 = v_2$$

Let v_1 = s, where s is any arbitrary nonzero constant.

Therefore the solution to this system is given by

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} s \\ s \\ 0 \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

From the above solution, it can be concluded that the vector

$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvector of the system associated with the eigenvalue $\lambda = 5$.

Usually the eigenvectors are normalized and the resulting vector represents the normalized eigenvector of the defined system.

To normalize a vector, determine its magnitude by taking the square root of the square of its components and divide each element by the magnitude. The resulting vector represents the normalized vector.

For example, the magnitude of the above-defined eigenvector is given by

$$mag = \sqrt{(1)^2 + (1)^2 + (0)^2} = \sqrt{2}$$

Therefore the normalized eigenvector of the matrix **A** is given by

$$\mathbf{n_1} = \frac{1}{mag}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{mag}\\ \frac{1}{mag}\\ \frac{0}{mag}\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\\ \frac{0}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}0.7071\\0.7071\\0\end{bmatrix}.$$

Also note that, any nonzero scalar multiple of the normalized eigenvector is also an eigenvector of the system.

Similar procedure must be followed to obtain the normalized eigenvector associated with the eigenvalue $\lambda = 0$.

Using MATLAB to calculate eigenvalues and eigenvectors

The eigenvalues and eigenvectors can be calculated in MATLAB using the "*eig*" command. By typing the command, "*help eig*" at the prompt in the MATLAB command window displays the following help message.

```
help eig

 EIG    Eigenvalues and eigenvectors.
    E = EIG(X) is a vector containing the eigenvalues of a square
    matrix X.

    [V,D] = EIG(X) produces a diagonal matrix D of eigenvalues and a
    full matrix V whose columns are the corresponding eigenvectors so
    that X*V = V*D.

    [V,D] = EIG(X,'nobalance') performs the computation with balancing
    disabled, which sometimes gives more accurate results for certain
    problems with unusual scaling.

    E = EIG(A,B) is a vector containing the generalized eigenvalues
    of square matrices A and B.

    [V,D] = EIG(A,B) produces a diagonal matrix D of generalized
    eigenvalues and a full matrix V whose columns are the
    corresponding eigenvectors so that A*V = B*V*D.
```

Example 3: Consider the following 3 x 3 matrix **A**

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 \\ 2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

The program to calculate the eigenvalues and eigenvectors of the defined matrix is given below.

```
% This program calculates the eigenvalues and eigenvectors.

A=[2 3 4;2 3 0;0 0 5];

[v,d]=eig(A)
```

Note that the first line of the program starts with a '%' sign. This represents that any character following the symbol is a comment and will not be included in the program while compiling.

Running the above program, gives the following result in the MATLAB command window.

```
v =

   -0.8321   -0.7071    0.7071
    0.5547   -0.7071    0.7071
         0         0    0.0000


d =

     0     0     0
     0     5     0
     0     0     5
```

'd' is a diagonal matrix, whose diagonal elements represent the eigenvalues of the system

From the above result it can be seen that, for the eigenvalue $\lambda = 5$, the corresponding eigenvectors are given by the second and the third columns of the 'v' matrix. The reason for the existence of two eigenvectors is that, the eigenvalue $\lambda = 5$ is a double root.

Properties of eigenvalues and eigenvectors

1. Eigenvalues and eigenvectors are defined only for square matrices.

2. According to the definition, the zero vector cannot be an eigenvector. However the real number 0 can be an eigenvalue of a matrix.

3. Every eigenvalue has an infinite number of eigenvectors associated with it, as any nonzero scalar multiple of an eigenvector is also an eigenvector.

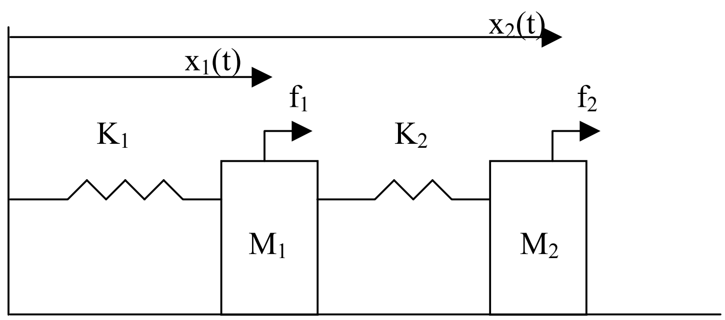
4. A matrix is invertible if and only if none of its eigenvalues is equal to zero.

Importance of eigenvalues and eigenvectors

There are various reasons, for calculating eigenvalues and eigenvectors. This handout outlines one such reason.

Vibrations point of view

Consider the system given below. It consists of two masses interconnected by a spring and forced to move on a horizontal plane.



The governing differential equation of motion for the above-defined system is

$$[\mathbf{M}]\ddot{x}(t)+[\mathbf{K}]x(t)=[\mathbf{f}]$$

'**M**' is the mass matrix, '**K**' is the stiffness matrix and '**x**' is the time dependent vector whose elements represent the displacement of the two masses from the fixed support and '***f***' is the constant force matrix.

The eigenvalues of the system represent the *square* of the **natural frequencies** with which the system will vibrate and the eigenvectors represent the **mode shapes**. In other words, the eigenvectors denote how the masses vibrate with respect to each other. This is explained in detail with the help of an example.

Example 5: For the above defined system the mass matrix '**M**' and the stiffness matrix '**K**' are given by

$$\mathbf{M}=\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}; \qquad \mathbf{K}=\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}; \qquad \mathbf{f}=\begin{bmatrix}1\\ 0\end{bmatrix}$$

Eigenvalues (or in this case, the natural frequencies) and the eigenvectors are properties of the unforced system. In other words, the force, '*f*' matrix is not used in the calculation of the eigenvalues and eigenvectors. To calculate the eigenvalues and eigenvectors, the "*eig*" command in MATLAB is used. The code for the procedure is shown below.

```
m=[1 0; 0 2];
k=[2 -1;-1 1];
[v,d]=eig(k,m)
w_nat = sqrt(d)
```

Note that the way the "*eig*" command is used is different from that used in the previous cases. Always make sure that, the first argument in the "*eig*" command is the stiffness matrix and then the mass matrix. The diagonal elements of 'd' matrix represent the eigenvalues and the columns of the 'v' matrix represent the corresponding eigenvectors. The natural frequency of the system is the square root of the eigenvalues.

The result of the above code is

```
v =

    0.9628    0.4896
   -0.2703    0.8719


d =

    2.2808         0
         0    0.2192



w_nat =

    1.5102         0
         0    0.4682
```

From the above result, it can concluded that the natural frequencies of the system are 1.5102 and 0.4682 rad/s. The eigenvectors associated with the natural frequencies are $\mathbf{v_1} = \begin{bmatrix} 0.9628 \\ -0.2703 \end{bmatrix}$ *and* $\mathbf{v_2} = \begin{bmatrix} 0.4896 \\ 0.8719 \end{bmatrix}$ respectively. The eigenvectors represent, how the two masses vibrate with respect to each other.

For example, if the system is vibrating at 1.5102 rad/s, which is one of the natural frequencies, then from the eigenvector associated with this frequency, it can be concluded that the masses vibrate in the opposite direction i.e. they are out of phase. In other words, if mass M_1 moves a distance of 0.9628 units in one direction, then the mass M_2 moves a distance of 0.2703 units in the opposite direction.

Similarly if the system vibrates at 0.4682 rad/s, then both the masses vibrate in phase with each other i.e., in the same direction as can be seen from the same sign of the elements of the eigenvector.

Note: The eigenvalues are also used to determine the stability of the system. This is discussed in detail in a later handout.

Assignment

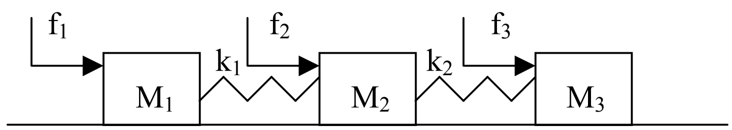
1) For the following matrices determine the eigenvalues and eigenvectors analytically and also verify the result using MATLAB.

$$\mathbf{A} = \begin{bmatrix} 4 & 9 \\ 2 & 7 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 4 & 3 & -1 \\ -2 & 0 & 2 \\ 3 & 3 & 0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 4 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

2) Consider the following system. Derive the governing equation of motion and obtain the eigenvalues and eigenvectors. Discuss in detail the mode shapes of the system i.e. explain in detail how the masses move with respect to each other when the system is excited with one of the natural frequencies. Explain the same for all the natural frequencies. (Note: The surface is frictionless.)



Given:

M_1 = 1Kg;
M_2 = 2Kg;
M_3 = 3Kg;

$k_1 = k_2$ = 10N/m;

The force matrix is defined as

$$\mathbf{F} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} N.$$

Recommended reading

"Feedback Control of Dynamic Systems" Third Edition, by Gene F. Franklin et.al – pp - 752.