Texas A & M University
Department of Mechanical Engineering
MEEN 364 Dynamic Systems and Controls
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Lecture 1: Modeling of Translational Mechanical Systems

The objective of this lecture is to review the basic building blocks of lumped parameter translational mechanical systems and to build the foundations that will enable you to model more complex dynamic systems.

Translational Inertia Elements (or Masses):

Analysis of mechanical systems is based on Newton's laws of motion. Usually, ideal (or point) masses are considered in the analysis, as shown in the Figure 1. Motion is considered with respect to a non-accelerating reference frame, usually a fixed point on earth or another non-accelerating object.

The equation of motion for a mass $m(t)$ is based on Newton's second law which expresses the conservation of linear momentum, as follows:

$$\frac{d(m(t)v_{1g}(t))}{dt} = \sum_{i=1}^{n} F_i(t), \tag{1}$$

where if we assume that the mass is constant m, we can rewrite this equation as

$$m(\frac{dv_{1g}(t)}{dt}) = \sum_{i=1}^{n} F_i(t) = F_m(t), \tag{2}$$

where $v_{1g}(t)$ is the velocity of the mass relative to the ground reference (g) and $F_m(t)$ is the *net* force acting on the mass.

Because of the non-accelerating nature of the reference frame

$$\frac{dx_{1g}}{dt} = \frac{dx_1}{dt}, \tag{3}$$

resulting in the following equation of motion for the ideal mass m

$$m(\frac{d^2x_1(t)}{dt^2}) = F_m(t). \tag{4}$$

Furthermore, the action of the applied force represents work being done on the mass as it accelerates, increasing its kinetic energy. The rate at which energy is stored in the system is equal to the rate at which work is expended on it. Using the first law of thermodynamics (or the law of conservation of energy) we have

$$\frac{d\boldsymbol{\mathcal{E}}_K(t)}{dt} = F_m(t)v_{1g}(t). \tag{5}$$

To find the energy of the system we need to integrate over a period $[0, t]$, as follows:

$$\int_{\boldsymbol{\mathcal{E}}_K(0)}^{\boldsymbol{\mathcal{E}}_K(t)} d\boldsymbol{\mathcal{E}}_K = \int_0^t F_m(t)v_{1g}(t)dt = \int_0^t mv_{1g}(t)(\frac{dv_{1g}(t)}{dt})dt = m\int_{v_{1g}(0)}^{v_{1g}(t)} v_{1g}(t)dv_{1g}, \tag{6}$$

or

$$\boldsymbol{\mathcal{E}}_K(t) = \boldsymbol{\mathcal{E}}_K(0) + (\frac{m}{2})v_{1g}^2(t). \tag{7}$$

This is the well-know formula for the kinetic energy of a point mass.

Equation (4) indicates that because of the integrations involved, it takes some time for the moving object to build-up velocity and displacement. As such, it would not be realistic to attempt to apply a step change in *velocity* (or *displacement*) of the mass. This would require an infinite amount of force and an infinite source of power!

Translational Stiffness Elements (or Springs):

An ideal translational spring, stores potential energy as it is deflected along its axis. This is depicted in Figure 2. The figure shows a spring in its relaxed state, $F_k = 0$, and with the force F_k acting at both ends, in free-body diagram fashion. Because an ideal spring has no mass, the force transmitted by it is

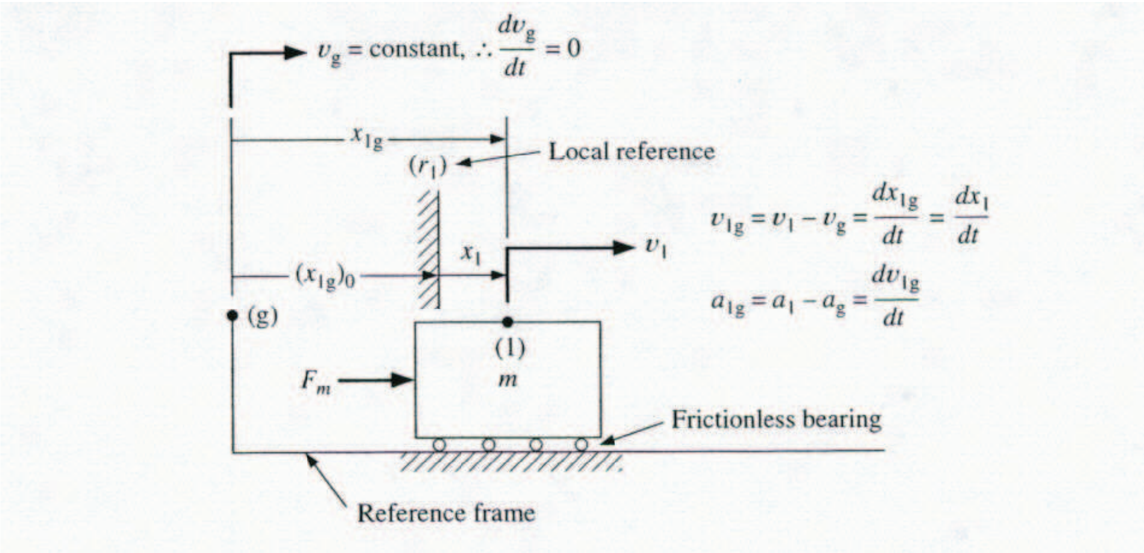

Figure 1: Free-body diagram of an ideal mass.

undiminished during acceleration. Therefore, the forces acting on its ends must be equal and opposite (Newton's third law of motion). The elemental equation for such a spring derives from Hooke's law, namely

$$F_k(t) = k[x_{21}(t) - (x_{21})_0], \tag{8}$$

where $(x_{21})_0$ is the free length of the spring. Because $x_{21}(t) - (x_{21})_0 = x_2(t) - x_1(t)$ we can write the equation for a spring as

$$F_k(t) = k[x_2(t) - x_1(t)], \tag{9}$$

where $x_2(t) - x_1(t)$ is the deflection of the spring from its initial free length.

In this development we have assumed that the spring has a constant stiffness, k. If this is not the case, we can write the general form of equation (9) as

$$F_{NLS}(t) = f_{NL}(x_2(t) - x_1(t)), \tag{10}$$

where NL stands for a non-linear spring. Simplification of this equation, via linearization, to a spring with an equivalent linear stiffness near an operating point will be discussed in future lectures.

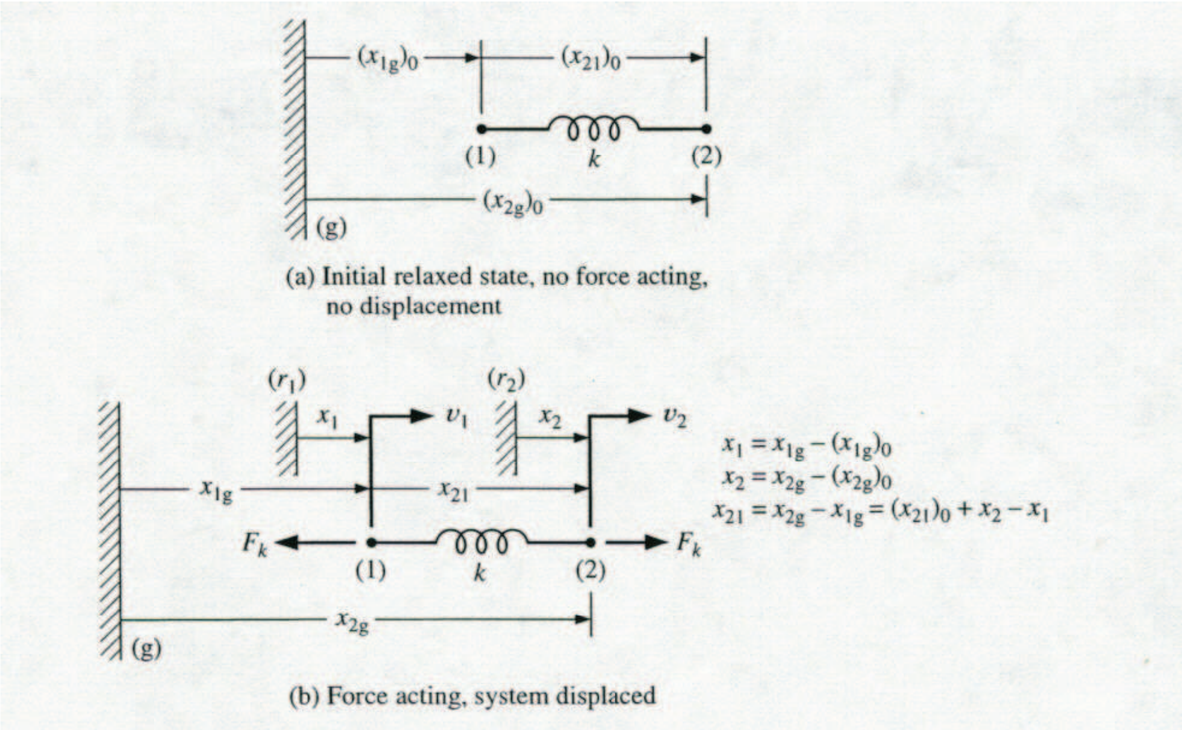

Figure 2: Free-body diagram of an ideal spring.

Similar to the case of an ideal mass, we can investigate the energy point-of-view of an ideal spring. The conservation of energy for an ideal spring can be expressed as

$$\frac{d\boldsymbol{\mathcal{E}}_P(t)}{dt} = F_k(t)v_{1g}(t). \tag{11}$$

To find the energy of the system we need to integrate as follows:

$$\int_{\boldsymbol{\mathcal{E}}_P(0)}^{\boldsymbol{\mathcal{E}}_P(t)} d\boldsymbol{\mathcal{E}}_P = \int_0^t F_k(t)v_{1g}(t)dt = (\frac{1}{k})\int_0^t F_k(t)(\frac{dF_k(t)}{dt})dt = (\frac{1}{k})\int_{F_k(0)}^{F_k(t)} F_k(t)dF_k, \tag{12}$$

or

$$\boldsymbol{\mathcal{E}}_P(t) = (\frac{1}{2k})F_k^2(t). \tag{13}$$

This is the formula for the potential energy of an ideal spring.

As in the case of the point mass, it would not be realistic to attempt to apply a step change in *force* to a spring. Such a force would have to move infinitely fast to deflect the spring suddenly, which would require an infinite

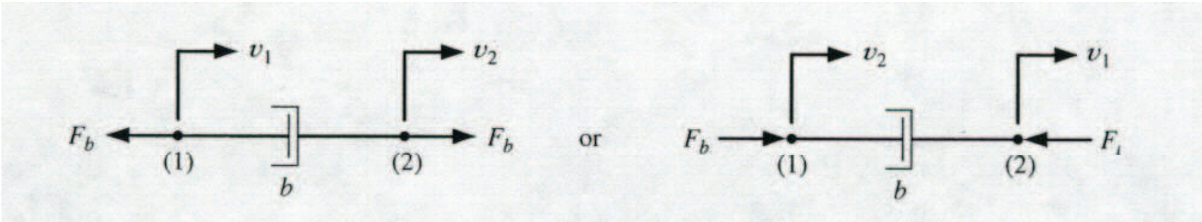

Figure 3: Free-body diagram of an ideal damper.

source of power!

Translational Damping Elements (or Dampers):

An ideal damper is shown in Figure 3. Because an ideal damper contains no mass, the force transmitted through it is undiminished during acceleration. Therefore, the forces acting at its ends must always be equal and opposite. The basic equation of an ideal damper is

$$F_b(t) = b(v_{2g}(t) - v_{1g}(t)) = bv_{21}(t). \tag{14}$$

With a damper there is no storage of retrievable mechanical work, as the work being done by an applied force becomes dissipated as thermal internal energy. The relationship between the force and velocity is instantaneous. Thus, it is realistic to apply step changes of either *force* or *velocity* to such an element.

An ideal damper arises from viscous friction between well-lubricated moving mechanical parts of a system. This is the only form of damping that is linear. Non-ideal forms of damping are very common in practice. However, non-ideal damping is characterized by non-linearities, such as dry (Coulomb) friction, aerodynamic damping, structural damping. These forms of damping can be linearized for simplification when no discontinuities exist in the force-velocity characteristics of the damper, at the expense of accuracy of the analysis. Non-ideal damping is found, for example, in poorly lubricated metal-metal contact surfaces.

In general, for a nonlinear damper we can write

$$F_{NLD}(t) = f_{NL}(v_{12}(t)), \tag{15}$$

where NL stands for a non-linear damper. Simplification of this equation, via linearization, to a damper with equivalent linear damping near an operating point will be discussed in future lectures.

Example: A Mass-spring-damper System in a Gravity Field:

The system to be analyzed is shown in Figure 4. We apply Newton's second law to the mass m, resulting in

$$F_i(t) + mg - F_k(t) - F_b(t) = m\frac{dv_{1g}(t)}{dt}. \tag{16}$$

The equations for the damper and the spring can be expressed as

$$F_b(t) = bv_{1g}(t), \tag{17}$$

and

$$F_k(t) = k(x_1(t) + \Delta_1 - x_g). \tag{18}$$

Equations (16) through (18) can be combined to obtain the following equation of motion

$$m(\frac{d^2x_1(t)}{dt^2}) + b(\frac{dx_1(t)}{dt}) + k(x_1(t) - x_g) = F_i(t), \tag{19}$$

where the $k\Delta_1$ term was eliminated using the mg term. With $x_g = 0$, we obtain the following second order equation

$$m(\frac{d^2x_1(t)}{dt^2}) + b(\frac{dx_1(t)}{dt}) + kx_1(t) = F_i(t). \tag{20}$$

This second order system (also called a 1 degree-of-freedom or 1DOF system) requires two variables to be completely described, velocity and position. We

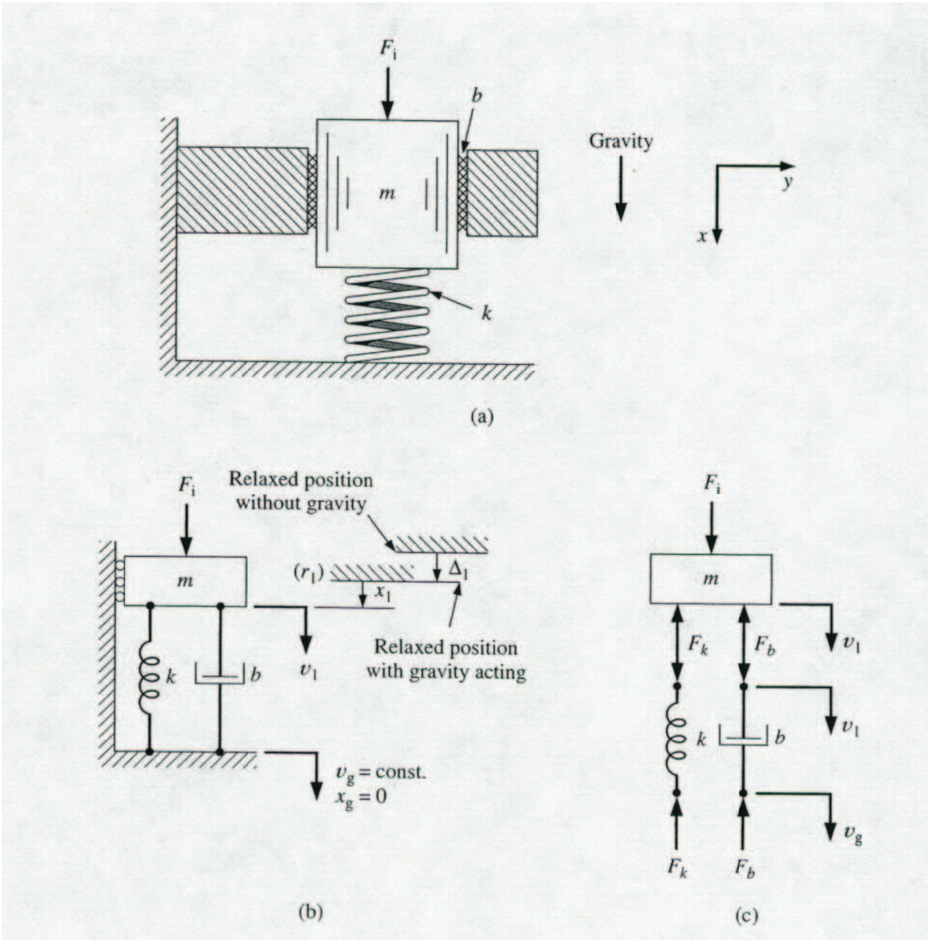

Figure 4: Diagram of a mass-spring-damper system in a gravity field.

call these variables the states of the system. This system also has two energy storage devices, the mass and the spring, exchanging kinetic and potential energy. The damper is an energy dissipation element.

Reading Assignment

See separate file on textbook reading assignments depending on the text edition you own. Read the examples Handout E.5 posted on the course web page.