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MEEN 364 Dynamic Systems and Controls
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Lecture 10: Laplace Transforms

The objective of this lecture is to introduce the concepts and mathematics involved in the Laplace transform (LT) and its use in dynamic systems and controls. LTs will be instrumental in the development of transfer functions and they will be used throughout this course in the analysis of dynamic systems and in the design of control systems.

Definition of the Laplace Transform and the Inverse Laplace Transform

The LT allows us to transform functions of time into functions of a complex variable s and it is a generalization of the Fourier Transform we discussed last week. It is defined as

$$\mathbf{F}(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt, \tag{1}$$

where s is a complex variable, $s = \sigma + j\omega$, and $f(t)$ is a continuous function of time[1]. For the integral in equation (1) to be computable there must exist some real numbers A and b, such that $|f(t)| < Ae^{bt}$. Most functions encountered in engineering are of this form.

The most common use of LTs is to solve ODEs. By using LTs we can transform ODEs form the t domain into algebraic equations in the s domain. The algebraic equations are typically easier to solve that the equivalent ODEs. The solutions of the algebraic equations can then be transformed back to the time domain using the inverse LT defined by the integral

$$f(t) = \mathcal{L}^{-1}\{\mathbf{F}(s)\} = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \mathbf{F}(s)e^{st}ds. \tag{2}$$

[1]Furthermore, the function $f(t)$ is assumed zero for $t < 0$ and the resulting transform is called one-sided LT

Table 1: Laplace Transforms of Most Common Functions of Time

Continuous Function	Laplace Transform
Impulse	1
Step	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^2	$\frac{2}{s^3}$
e^{-at}	$\frac{1}{(s+a)}$
te^{-at}	$\frac{1}{(s+a)^2}$
$sin(\omega t)$	$\frac{\omega}{(s^2+\omega^2)}$
$cos(\omega t)$	$\frac{s}{(s^2+\omega^2)}$

Typically the definition of the inverse LT is not used in practice. Rather, the method of partial fraction expansion is used to obtain the inverse LT, as it will be shown in the examples.

Table 1 below summarizes the LTs of the most commonly used time functions.

Basic Properties of the Laplace Transform

The LT has a number of useful properties that are handy in manipulating transfer functions of dynamic systems. The most important properties of the LT are listed below.

(1) Linearity

$$\mathcal{L}\{a_1 f_1(t) + a_2 f_2(t)\} = a_1 \mathbf{F}_1(s) + a_2 \mathbf{F}_2(s). \tag{3}$$

(2) Integration

$$\mathcal{L}\{\int_0^t f(\tau)d\tau\} = \frac{\mathbf{F}(s)}{s}. \tag{4}$$

(3) Differentiation

$$\mathcal{L}\{\frac{d^n f(t)}{dt^n}\} = s^n \mathbf{F}(s) - \sum_{k=0}^{n-1} s^{n-k-1} [\frac{d^k f(t)}{dt^k}]|_{t=0^-}. \tag{5}$$

(4) Shifting in the time domain

$$\mathcal{L}\{f(t-a)\} = e^{-as}\mathbf{F}(s). \tag{6}$$

(5) Shifting in the complex domain

$$\mathcal{L}\{f(t)e^{-at}\} = \mathbf{F}(s+a). \tag{7}$$

Additionally, the initial and final value theorems associated with the LT find very useful applications. The initial value theorem can be expressed as

$$f(0^+) = \lim_{s\to\infty} s\mathbf{F}(s). \tag{8}$$

The final value theorem allows calculation of the steady-state value of a function as follows

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\mathbf{F}(s). \tag{9}$$

Partial Fraction Expansion Method

See Handout A.2.

An Example

Consider the mass-spring-dashpot system shown in Figure 1. Use the LT to find the system transfer function from the forcing function $F(t)$ to the displacement $x(t)$.

The equations of motion for this system are

$$m\frac{dv}{dt} = F(t) - bv(t) - kx(t), \tag{10}$$

and

$$\frac{dx(t)}{dt} = v(t), \tag{11}$$

Taking the LT of equations (10) and (11) results in

$$m[s\mathbf{V}(s) - v(0^-)] = \mathbf{F}(s) - b\mathbf{V}(s) - k\mathbf{X}(s), \tag{12}$$

and

$$s\mathbf{X}(s) - x(0^-) = \mathbf{V}(s), \tag{13}$$

where $x(0^-)$ and $v(0^-)$ are the initial conditions for displacement and velocity of mass m just before the force is applied. Now, eliminating $\mathbf{V}(s)$ from

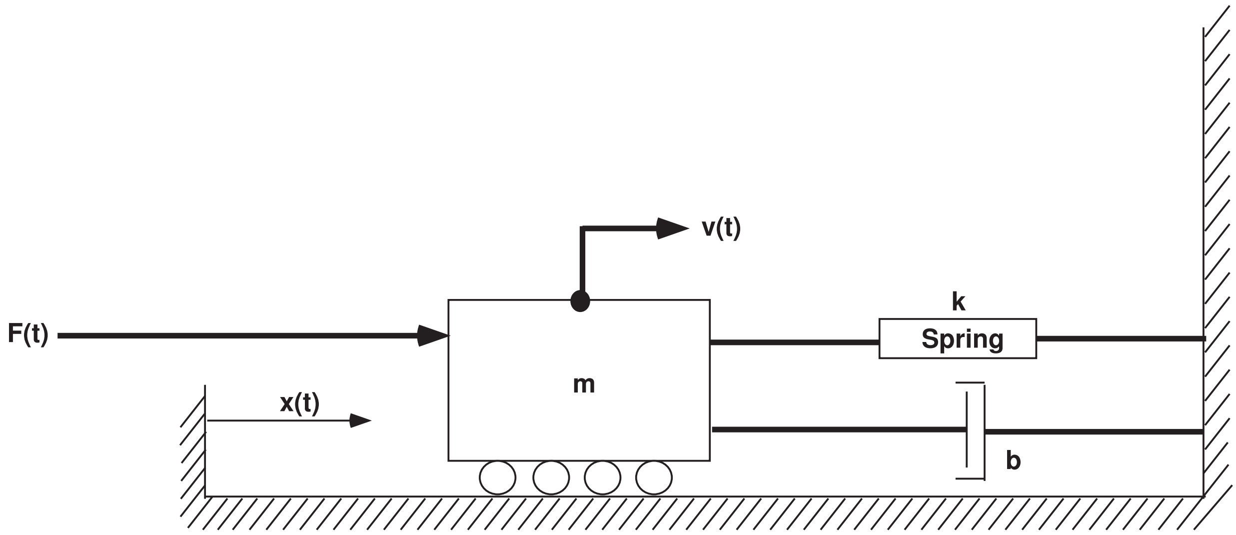

Figure 1: Schematic Diagram of Mass-Spring-Dashpot System.

equations (12) and (13) results in

$$\mathbf{X}(s)(ms^2 + bs + k) = \mathbf{F}(s) + x(0^-)(ms + b) + mv(0^-). \tag{14}$$

Assuming zero initial conditions, the system transfer function is found to be

$$\mathbf{T}(s) = \frac{\mathbf{X}(s)}{\mathbf{F}(s)} = \frac{1}{ms^2 + bs + k}. \tag{15}$$

Reading Assignment

Read "Handout A.2: Laplace Transforms" and the examples Handout E.3 posted on the course web page.