Texas A & M University
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MEEN 364 Dynamic Systems and Controls

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Lecture 11: Transfer Functions & Block Diagrams
Poles and Zeros of Transfer Functions

The objective of this lecture is to introduce you to the basic concepts of impulse response, convolution integral, transfer functions and block diagrams. These are the very basic tools for the analysis of linear time-invariant (or constant) systems.

Convolution and Dynamic Response of Linear Systems

The principle of superposition states that if a system has an input that can be expressed as a sum of signals, then the response of the system can be expressed as the same sum of the individual responses to the respective signals. For example, consider a system with input $u(t)$ and output $y(t)$ and initially at rest. We apply the input $u_1(t)$ and $u_2(t)$ and observer the responses $y_1(t)$ and $y_2(t)$, respectively. Then we form the input $u(t) = \alpha_1\ u_1(t) + \alpha_2\ u_2(t)$. If superposition applies then the system response will be $y(t) = \alpha_1\ y_1(t) + \alpha_2\ y_2(t)$. Superposition is valid only for linear systems.

A simple way to obtain the response of a linear system is to write down the so-called "convolution integral". The convolution integral is the result of applying the principle of superposition (to a linear system) for the specific case when the input signals are impulses.

Let us first define an impulse signal. An impulse is a very intense short duration signal denoted by $\delta(t)$, which has the property that is a function

$f(t)$ is continuous at $t = \tau$, then

$$\int_{-\infty}^{+\infty} f(t)\delta(t-\tau)d\tau = f(t). \tag{1}$$

This indicates that an impulse is so intense that no other value of $f(t)$ matters expect over the short period over which the impulse occurs. If we replace $f(t)$ by $u(t)$, then equation (1) represents an input $u(t)$ as a sum of impulses of intensity $u(t-\tau)$.

If for a general linear system we express the impulse response as $h(t,\tau)$, the response at t to an impulse applied at τ, then the total response of a system to an input $u(t)$ can be expressed as

$$y(t) = \int_{-\infty}^{+\infty} u(\tau)h(t,\tau)d\tau. \tag{2}$$

This is the superposition integral applied only to linear systems. If the system is also time-invariant (constant) then its impulse response is given by $h(t-\tau)$ because the response of the system at t to an input at τ depends on the difference between these two times. In other words

$$y(t) = \int_{-\infty}^{+\infty} u(\tau)h(t-\tau)d\tau, \tag{3}$$

or

$$y(t) = \int_{-\infty}^{+\infty} u(t-\tau)h(\tau)d\tau = u(t) * h(t). \tag{4}$$

Equation (4) represents the convolution integral.

Transfer Functions

The most immediate consequence of the convolution integral is the concept of transfer function. If a linear system has an input of the form e^{st}, where $s = \sigma + j\omega$, then the output of that system will be of the form $H(s)e^{st}$. Both input and output are of exponential form and they differ by the amplitude $H(s)$, which is the system transfer function.

If we let $u(t) = e^{st}$ in equation (4) then

$$y(t) \;=\; \int_{-\infty}^{+\infty} u(t-\tau)h(\tau)d\tau$$

$$
\begin{aligned}
&= \int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}d\tau \\
&= \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau e^{st} \\
&= H(s)e^{st} \qquad (5)
\end{aligned}
$$

This is the exponential response of the system. To find the transfer function one does not have to compute the integral in equation (5). Rather, extensive use of the properties of the Laplace transform is made.

A common way to use the exponential response of a linear time-invariant system is in finding its frequency response, or its response to a sine input. First we have that

$$A cos(\omega t) = \frac{A}{2}(e^{j\omega t} + e^{-j\omega t}). \qquad (6)$$

The response of the system to input $u(t) = e^{\pm j\omega t}$ is $y(t) = H(\pm j\omega)e^{\pm j\omega t}$. From superposition, we have that the response of the system to a cosine is

$$y(t) = \frac{A}{2}[H(j\omega)e^{j\omega t} + H(-j\omega)e^{-j\omega t}]. \qquad (7)$$

If we represent the complex number H in polar form $Me^{j\phi}$, we have

$$
\begin{aligned}
y(t) &= \frac{A}{2}M(e^{j(\omega t+\phi)} + e^{-j(\omega t+\phi)}) \\
&= AM cos(\omega t + \phi). \qquad (8)
\end{aligned}
$$

We can generalize this result to more than just *cos* inputs buy taking the Laplace transforms of the convolution integral, equation (4). Then we find that

$$Y(s) = H(s)U(s), \qquad (9)$$

Using Laplace Transforms to Solve Problems

Laplace Transforms and transfer functions can be used for solving differential equations.

For example, find the solution of the differential equation

$$\ddot{y}(t) + y(t) = 0, \qquad (10)$$

where

$$y(0) = \alpha,\ \dot{y}(0) = \beta. \tag{11}$$

Taking the Laplace transform of both side of the differential equation we have

$$s^2Y(s) - \alpha s - \beta + Y(s) = 0, \tag{12}$$

or, solving for $Y(s)$

$$Y(s) = \frac{\alpha s}{s^2+1} + \frac{\beta}{s^2+1}. \tag{13}$$

Taking the inverse Laplace transform of these two terms (using a table look-up) we have

$$y(t) = \alpha cos(t) + \beta sin(t). \tag{14}$$

In previous lectures we expressed the dynamics of various systems in state-variable (or state-space) form as

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t), \\ y(t) &= \mathbf{C}\mathbf{x}(t) + du(t). \end{aligned} \tag{15}$$

We also saw that a linear dynamic system can be expressed by its transfer function, or the Laplace transform of its differential equation. Given a state-space form of equation (15) we can obtain the following equivalent transfer function

$$H(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + d. \tag{16}$$

Note: Derive the expression of equation (16).

The transfer function $H(s)$ can be expressed in two forms. As the ratio of two polynomials in s

$$H(s) = \frac{b_1 s^m + b_2 s^{m-1} + \cdots + b_{m+1}}{s^n + a_1 s^{n-1} + \cdots + a_n}, \tag{17}$$

or in pole-zero form

$$H(s) = K\frac{\Pi_{i=1}^{m}(s - z_i}{\Pi_{i=1}^{n}(s - p_i)}. \tag{18}$$

In equation (18) the z_i's and p_i's are the zeros and poles of the transfer function $H(s)$, respectively.

Block Diagrams and Block Diagram Manipulation

When manipulating block diagrams a common configuration is that of **negative feedback loop**. The transfer resulting from this configuration is

$$Y_1(s) = \frac{G_1(s)}{1 + G_1(s)G_2(s)} R(s). \tag{19}$$

If we have **positive feedback**, then the denominator of equation (19) becomes $1 - G_1(s)G_2(s)$. If there is no component in the feedback path, the system is called **unity feedback system**.

Note: Read the block diagram manipulations of pages 123-127. Read example 3.20 on page 126.

Reading Assignment

See separate file on textbook reading assignments depending on the text edition you own. This will be a good review for you. Read examples Handout E.13 posted on the course web page.