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Lecture 12: Dynamic Response of Linear Systems Impact of Pole & Zero Locations

The objective of this lecture is to provide you with some background on the use of transfer function poles and zeros for determination of system dynamic response in the time-domain.

Time Response Versus Pole Locations

Once the transfer function of a dynamic system is calculated, it can almost always be expressed as the ratio of two polynomials. These polynomials can be used to compute the system poles and zeros which completely determine the system response, to within a constant. Poles and zeros can be used to determine the time history of a dynamic system, and relate time histories to the location of the poles and zeros on the s-plane.

The impulse response of a system is a time-function that corresponds to the transfer function of a system. As such, it is usually called the **natural response** of the system.

Why?

First Order Systems

For example, for a first-order pole

$$H(s) = \frac{1}{s+\sigma},\tag{1}$$



Figure 1: First order system response.

which corresponds to the following impulse response

$$h(t) = e^{-\sigma t} \mathbf{1}(t), \tag{2}$$

where 1(t) is the unit step function. For $\sigma > 0$ the pole is located on the left-half (LHP) of the complex plane (Re(s) < 0) and the exponential decays resulting in a **stable** impulse response. On the contrary, if $\sigma < 0$ the pole resides in the RHP and the impulse response is said to be **unstable**. Further, we define the time constant of the impulse response as

$$\tau = \frac{1}{\sigma},\tag{3}$$

as shown in Figure 1.

Examples of Transient Response Versus Real Pole Locations

Compare the time response and pole locations of the system with transfer function

$$H(s) = \frac{2s+1}{s^2+3s+2}.$$
(4)



Figure 2: Pole-zero locations of first order system.

The denominator can be written as (s+1)(s+2) and the poles of the system are at -1 and -2. The system has only one zero at $-\frac{1}{2}$. Performing a partial fraction expansion on the transfer function (4) results in

$$H(s) = -\frac{1}{s+1} + \frac{3}{s+2}.$$
(5)

Obtaining the inverse of the components results in the following impulse response function

$$h(t) = -e^{-t} + 3e^{-2t}; \ t \ge 0.$$
(6)

Equation (6) reveals that the shape of the impulse response depends on the pole locations, as shown in Figure 2.

In fact, this observation is more general. The shape of the natural response of a system depends on the location of the poles of the transfer function. This generalization is described in Figure 3. The role of the numerator (zeros) in the overall system response is that it influences the size of the coefficients multiplying each component of the response. Also, a "fast" pole is one that decays faster compared to another pole, which would be called a "slow" pole.

Second Order Systems



Figure 3: Time functions associated with pole locations.

A pole category that requires special attention is that of complex poles. These can be expressed in terms of their real and imaginary parts, or

$$s = -\sigma \pm j\omega_d. \tag{7}$$

Since there are always complex conjugate pairs of complex poles, the denominator corresponding to complex poles becomes

$$a(s) = (s + \sigma - j\omega_d)(s + \sigma + j\omega_d) = (s + \sigma)^2 + \omega_d^2.$$
(8)

However, it is common practice to write a second order transfer function as

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$
(9)

Comparing the denominator polynomial of equation (9) with equation (8) we conclude that

$$\sigma = \zeta \omega_n, \tag{10}$$



Figure 4: Second order system pole locations.

and

$$\omega_d = \omega_n \sqrt{1 - \zeta^2},\tag{11}$$

where ζ is the damping ratio and ω_n is the undamped natural frequency, and ω_d is the damped natural frequency. The pole locations for a second order transfer function are shown in Figure 4.

Equation (9) can be rewritten as

$$H(s) = \frac{\omega_n^2}{(s + \zeta \omega_n)^2 + \omega_n^2 (1 - \zeta^2)} = \frac{\omega_n^2}{(s + \sigma)^2 + \omega_d^2},$$
(12)

which corresponds to an impulse response function of

$$h(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\sigma t} \sin(\omega_d t) \mathbf{1}(t).$$
(13)

<u>Note:</u> Prove this to yourselves.

The time response of a second order system for different values of the damping ratio are shown in the top segment of Figure 5. One should note that as the damping ratio increases the actual frequency ω_d decreases slightly. For very low values of damping (ζ close to zero) the response is oscillatory, while for large values of damping (ζ close to one) the response shows no oscillations.



Figure 5: Second order system response for different damping (a) impulse responses; (b) step responses.



Figure 6: Pole locations corresponding to different damping ratios.

Three complex pairs of poles are shown in Figure 6, corresponding to different values of the damping ratio. The imaginary part of the poles determines the damped natural frequency, which in this case remains the same. This despite changes in the (undamped) natural frequency and damping ratio. The real part of the poles determines the decay rate of the exponential envelope. For $\sigma < 0$ the response decays, whereas for $\sigma > 0$ becomes unbounded (or unstable). For $\sigma = 0$ the response neither decays nor grows. Practically, the system is considered unstable.

Time-domain Specifications

In the design of a mechanical system, of any other type of system, or even a control system, we often specify requirements in terms of a system's time response. These requirements, shown in Figure 7, are defined as follows:

• rise time, t_r , the time it takes for the system to reach the vicinity of its new set-point.



Figure 7: Time-domain system specifications.

- settling time, t_s , the time it takes the system transients to decay,
- overshoot, M_p , the maximum system overshoot divided by its final value,
- **peak time**, t_p , the time it takes for a system to reach the maximum overshoot point.

For a second order system with no zeros the rise time can be approximately expressed as

$$t_r \cong \frac{1.8}{\omega_n}.\tag{14}$$

For the overshoot more precise expressions can be obtained. At the overshoot value, the derivative of the response is zero. The step response of a second order system in the time domain is given by

$$y(t) = 1 - e^{-\sigma t} (\cos\omega_d t + \frac{\sigma}{\omega_d} \sin\omega_d t),$$
(15)

where $\omega_d = \omega_n \sqrt{1 - \zeta^2}$ and $\sigma = \zeta \omega_n$. Taking the derivative of expression (15)

and setting it to zero we obtain

$$\dot{y}(t) = \sigma e^{-\sigma t} (\cos\omega_d t + \frac{\sigma}{\omega_d} \sin\omega_d t) - e^{-\sigma t} (-\omega_d \sin\omega_d t + \sigma \cos\omega_d t) = 0$$
$$= e^{-\sigma t} (\frac{\sigma^2}{\omega_d} \sin\omega_d t + \omega_d \sin\omega_d t) = 0.$$
(16)

The zero point occurs when

$$\omega_d t_p = \pi, \tag{17}$$

and we have the following formula for the peak time

$$t_p = \frac{\pi}{\omega_d}.\tag{18}$$

For this value of time, the overshoot is equal to

$$y(t_p) = 1 + M_p = 1 - e^{-\sigma\pi/\omega_d} (\cos\pi + \frac{\sigma}{\omega_d} \sin\pi),$$

= 1 + e^{-\sigma\pi/\omega_d}. (19)

So, we have the following formula for the overshoot

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}, \ 0 \le \zeta < 1.$$
 (20)

For a second order system, a plot of the overshoot M_p versus the damping ratio is shown in Figure 8.

Finally, we would like an expression for the settling time. We can define the settling time as that value of t_s when the decaying exponential reaches 1%. So,

$$e^{-\zeta\omega_n t_s} = 0.01,\tag{21}$$

or

$$t_s = \frac{4.6}{\zeta \omega_n} = \frac{4.6}{\sigma}.$$
(22)

In summary, for <u>a second order system with no zeros</u>, we have the following equations that determine its time-domain characteristics

$$\omega_n \geq \frac{1.8}{t_r},\tag{23}$$

$$\zeta \geq \zeta(M_p), \tag{24}$$

$$\sigma \geq \frac{4.0}{t_s}.$$
 (25)



Figure 8: Overshoot versus damping ratio for a second order system.

These inequalities can be graphed in the s-plane and used for design, as shown in Figure 9.

The Effects of Zeros and Additional Poles

So far we have only accounted for the time-domain characteristics of a second order system with no zeros. If the system has zeros or additional poles (higher than second order system) then the transient response of the system is more complex and it cannot be expressed in terms of simple equations, as before.

Mathematically speaking, the presence of zeros in a transfer function is to modify the coefficients of the exponential terms in the transient response. For example, consider two transfer functions with the same poles but different



Figure 9: Time-domain specifications in the s-plane.

zeros, as follows

$$H_{1}(s) = \frac{2}{(s+1)(s+2)}$$

$$= \frac{2}{s+1} - \frac{2}{s+2},$$

$$H_{2}(s) = \frac{2(s+1.1)}{1.1(s+1)(s+2)}$$

$$= \frac{2}{1.1}(\frac{0.1}{s+1} + \frac{0.9}{s+2})$$

$$= \frac{0.18}{s+1} + \frac{1.64}{s+2}.$$
(26)
(27)

The two transfer functions are normalized to have the same DC gain, i.e. the transfer function value at s = 0. Notice that the coefficient of the (s + 1) term has been reduced dramatically from 2 to 0.18. This reduction is the result of the zero at s = -1.1 which is close to the pole at s = -1. If the zero is place exactly where the pole is located, then this term will disappear.

To account for the effect of a zero on the overall system transient response, consider a transfer function with a zero and two complex poles, as follows:

$$H(s) = \frac{(s/\alpha\zeta\omega_n) + 1}{(s/\omega_n)^2 + 2\zeta(s/\omega_n) + 1}.$$
(28)

The zero for the transfer function is locate at $s = -\alpha \zeta \omega_n = -\alpha \sigma$. If α is large then the zero will be far away from the real part of the poles and its



Figure 10: Impact of zero location on transient response.

impact will be minimal. If $\alpha \cong 1$ then the zero will be close to the poles and its impact will be significant. The step response of such a system with 0.5 damping ratio and for different values of α is shown in Figure 10. We see that the major impact of the zero is to increase the overshoot M_p , with little impact on the settling time.

When the value of α is negative, then there is a zero on the RHP, also called a nonminimum-phase zero. The transient response of the resulting system is quite different. In fact, the overshoot is suppressed to the point that the response first starts in the wrong direction and then changes sign.

In studying the effect of an additional pole, let us consider the following transfer function

$$H(s) = \frac{1}{(s/\alpha\zeta\omega_n + 1)((s/\omega_n)^2 + 2\zeta(s/\omega_n) + 1)}.$$
(29)

Plots of the transient response of such a system with 0.5 damping ratio for different values of α are shown in Figure 11. In this case, the major effect



Figure 11: Impact of an additional pole on transient response.

is the increase in the rise time, i.e. the system response slows-down as the additional pole gets closer to the real part of the complex poles.

The Effects of Pole-Zero Patterns on Dynamic Response – A Summary

• For a <u>second order system with no zeros</u>, the transient response parameters can be approximately obtained by

$$Rise \ Time: \ t_r \cong \frac{1.8}{\omega_n},\tag{30}$$

$$Overshoot: M_p = M_p(\zeta), (see \ Figure \ 8)$$
(31)

Settling time:
$$t_s \cong \frac{4.6}{\sigma}$$
. (32)

<u>Note:</u> This kind of system is also said to have second order dominance.

• A zero in the LHP will increase the overshoot if the zero is within a factor of 4 of the real part of the complex poles, otherwise its impact is minimal.

- A zero in the RHP will depress the overshoot (and will cause the response to start in the wrong direction non-minimum phase response)
- A additional pole in the LHP will increase the rise time significantly, if the extra pole is within a factor of 4 of the real part of the complex poles, otherwise its impact is minimal.

Reading Assignment

See separate file on textbook reading assignments depending on the text edition you own. Read examples Handout E.15 posted on the course web page.