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Department of Mechanical Engineering
MEEN 364 Dynamic Systems and Controls
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Lecture 2: State-space Representation of Dynamic Systems

The objective of this lecture is to introduce you to the two distinct models used in representing dynamic systems in the time-domain; namely input-output models and state-space models. The procedure for obtaining the state-space representation of an input-output model is also presented.

Input-Output Models

In dealing with dynamic systems we define *inputs* and *outputs*. Inputs originate outside the system and are not directly dependent on what happens in the system. Outputs are chosen from the set of variables generated by the system as it is subjected to the input variables. The choice of the outputs is fairly arbitrary.

Consider the single-input, single-output dynamic system shown in Figure 1. For most systems we will encounter in this class, the relation between the input and the output signal can be represented by the following n th order differential equation:

$$f\left(y(t), \frac{dy(t)}{dt}, \dots, \frac{d^n y(t)}{dt^n}, u(t), \frac{du(t)}{dt}, \dots, \frac{du^m(t)}{dt^m}\right) = 0, \quad (1)$$

where $m \leq n$ for physically realizable systems, and where the function f is, in general, nonlinear.

For a linear, single input, single output system, equation (1) can be sim-

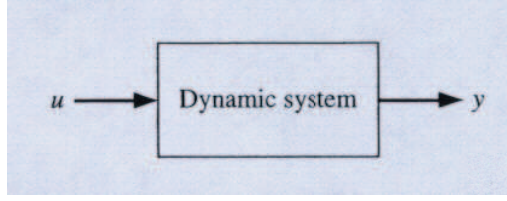


Figure 1: Single-input, single-output dynamic system.

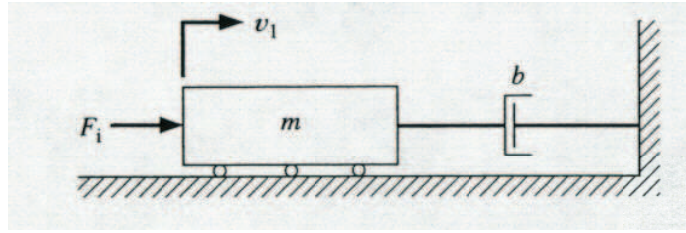


Figure 2: Mechanical system for example 1.

plified as

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \dots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = \quad (2)$$

$$b_m \frac{d^m u(t)}{dt^m} + b_{m-1} \frac{d^{m-1} u(t)}{dt^{m-1}} + \dots + b_1 \frac{du(t)}{dt} + b_0 u(t), \quad (3)$$

where $a_n, \dots, a_0$ and $b_m, \dots, b_0$ are all constant coefficients. Again, $m \leq n$.

Example 1

Derive an input-output model for the system shown in Figure 2. The mass m is supported by an oil film bearing that produces a resisting force proportional to the velocity of the mass.

For this system the choice of the input and output is rather obvious. The force $F_i(t)$ is the input and the resulting velocity $v_1(t)$ is the output. The system equation of motion is

$$m \frac{dv_1(t)}{dt} + bv_1(t) = F_i(t). \quad (4)$$

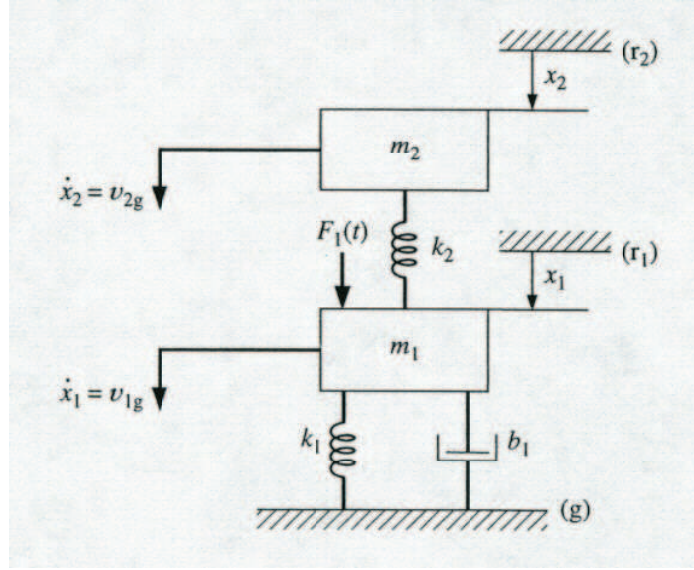


Figure 3: Mechanical system for example 2.

So, here $u(t) = F_i(t)$ and $y(t) = v_1(t)$, with $n = 1$ and $m = 0$. Also, $a_0 = b$, $a_1 = m$ and $b_0 = 1$.

Example 2

Derive the input-output equations for the mechanical system show in Figure 3, using the force $F_1(t)$ as the input variable and the displacements $x_1(t)$ and $x_2(t)$ as the output variables.

The equation of motion for mass m_1 is

$$m_1 \ddot{x}_1(t) + b_1 \dot{x}_1(t) + (k_1 + k_2)x_1(t) - k_2 x_2(t) = F_1(t). \quad (5)$$

The equation of motion for mass m_2 is

$$m_2 \ddot{x}_2(t) + k_2 x_2(t) - k_2 x_1(t) = 0. \quad (6)$$

Combining equations (5) and (6) and eliminating $x_1(t)$ yields the following input-output equation for the system

$$\frac{d^4 x_2}{dt^4} + \left(\frac{b_1}{m_1}\right) \frac{d^3 x_2}{dt^3} + \left(\frac{k_2}{m_2} + \frac{k_1}{m_1} + \frac{k_2}{m_1}\right) \frac{d^2 x_2}{dt^2}$$

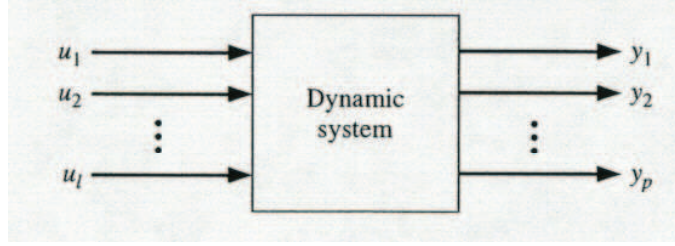


Figure 4: Multi-input, multi-output dynamic system.

$$+\frac{b_1 k_2}{m_1 m_2} \frac{dx_2}{dt} + \frac{k_1 k_2}{m_1 m_2} x_2 = \frac{k_2}{m_1 m_2} F_1(t). \quad (7)$$

This equation can be solved provided that four initial conditions and the input $F_1(t)$ is known. Similarly, the input-output equation relating $x_1(t)$ to $F_1(t)$ can be derived as

$$\begin{aligned} & \frac{d^4 x_1}{dt^4} + \left(\frac{b_1}{m_1}\right) \frac{d^3 x_1}{dt^3} + \left(\frac{k_2}{m_2} + \frac{k_1}{m_1} + \frac{k_2}{m_1}\right) \frac{d^2 x_1}{dt^2} \\ & + \frac{b_1 k_2}{m_1 m_2} \frac{dx_1}{dt} + \frac{k_1 k_2}{m_1 m_2} x_1 = \left(\frac{1}{m_1}\right) \frac{d^2 F_1}{dt^2} + \frac{k_2}{m_1 m_2} F_1(t). \end{aligned} \quad (8)$$

For multi-input, multi-output systems, equations (1), (2) and (3) can be generalized for use. Figure 4 depicts the block diagram of such a system.

State-Space Models

The reason we introduce state-space models, in addition to input-output models, is the fact that the former are much more powerful than the latter, and they are widely used in modeling complex engineering systems. The concept of a *state* is similar to that defined in thermodynamics. That is, state variables constitute the minimum number of variables which, if known, completely describe the system under consideration. When the state variables are grouped together they form the so-called *state vector*. The models that result from the use of the state vector are called state-space models. Finally, we define *state trajectory* as the path over time, followed by the state of a

system.

Mathematically, the state-space equations are sets of first-order differential equations. For a linear system model the state-space equations take the following form

$$\begin{aligned}\dot{q}_1(t) &= a_{11}q_1(t) + a_{12}q_2(t) + \dots + a_{1n}q_n(t) + b_{11}u_1(t) + b_{12}u_2(t) + \dots + b_{1m}u_m(t) \\ &\vdots\end{aligned}\tag{9}$$

$$\dot{q}_n(t) = a_{n1}q_1(t) + a_{n2}q_2(t) + \dots + a_{nn}q_n(t) + b_{n1}u_1(t) + b_{n2}u_2(t) + \dots + b_{nm}u_m(t),\tag{10}$$

where $q_1(t), \dots, q_n(t)$ are the state variables (also denoted by $x(t)$), and $u_1(t), \dots, u_m(t)$, are the input variables.

The system output equations are

$$\begin{aligned}y_1(t) &= c_{11}q_1(t) + c_{12}q_2(t) + \dots + c_{1n}q_n(t) + d_{11}u_1(t) + d_{12}u_2(t) + \dots + d_{1m}u_m(t) \\ &\vdots\end{aligned}\tag{11}$$

$$y_p(t) = c_{p1}q_1(t) + c_{p2}q_2(t) + \dots + c_{pn}q_n(t) + d_{p1}u_1(t) + d_{p2}u_2(t) + \dots + d_{pm}u_m(t),\tag{12}$$

$y_1(t), \dots, y_p(t)$ are the output variables.

These equations can be written in more compact matrix form as

$$\dot{\mathbf{q}}(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{B}\mathbf{u}(t),\tag{13}$$

and

$$\mathbf{y}(t) = \mathbf{C}\mathbf{q}(t) + \mathbf{D}\mathbf{u}(t),\tag{14}$$

where $\mathbf{A}$ is the $n \times n$ state matrix, $\mathbf{B}$ is the $n \times m$ input, $\mathbf{C}$ is the $p \times n$ output matrix, and $\mathbf{D}$ is the $p \times m$ direct output (or feedforward) matrix. Furthermore, $\mathbf{q}(t)$ is the state vector, $\mathbf{u}(t)$ is the input vector, and $\mathbf{y}(t)$ is the output vector.

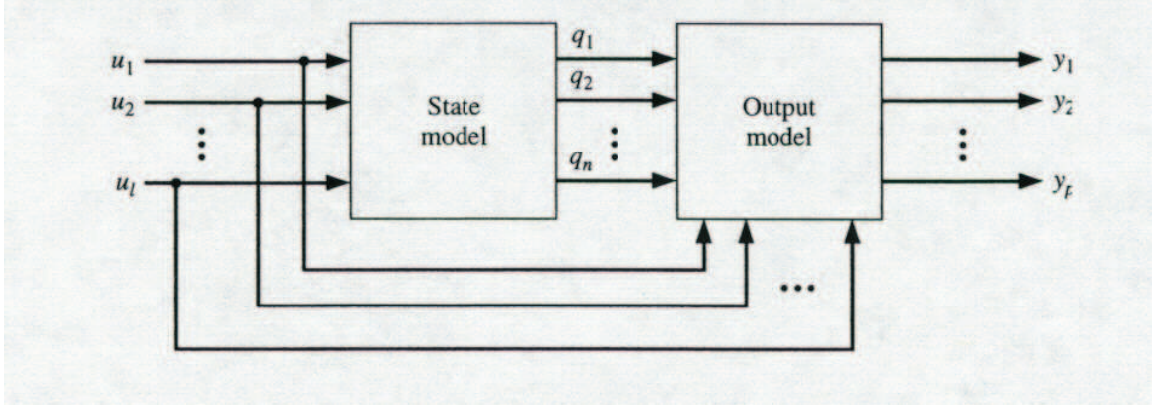


Figure 5: Block diagram of a state model.

In the case of a general nonlinear system, the state-space equations can be generalized as follows:

$$\dot{q}_1(t) = f_1(q_1(t), q_2(t), \dots, q_n(t), u_1(t), u_2(t), \dots, u_m(t)) \quad (15)$$

$$\vdots$$

$$\dot{q}_n(t) = f_n(q_1(t), q_2(t), \dots, q_n(t), u_1(t), u_2(t), \dots, u_m(t)), \quad (16)$$

whereas the output equations can be expressed as

$$y_1(t) = g_1(q_1(t), q_2(t), \dots, q_n(t), u_1(t), u_2(t), \dots, u_m(t)) \quad (17)$$

$$\vdots$$

$$y_p(t) = g_n(q_1(t), q_2(t), \dots, q_n(t), u_1(t), u_2(t), \dots, u_m(t)). \quad (18)$$

A block diagram of the state-space representation of this multi-input, multi-output nonlinear system is depicted in Figure 5.

Input-Output to State-space Transition

Input-output and state-space models are equivalent. As a result an input-output model can be transformed to a state-space model, and vice versa (though the latter is a bit more cumbersome).

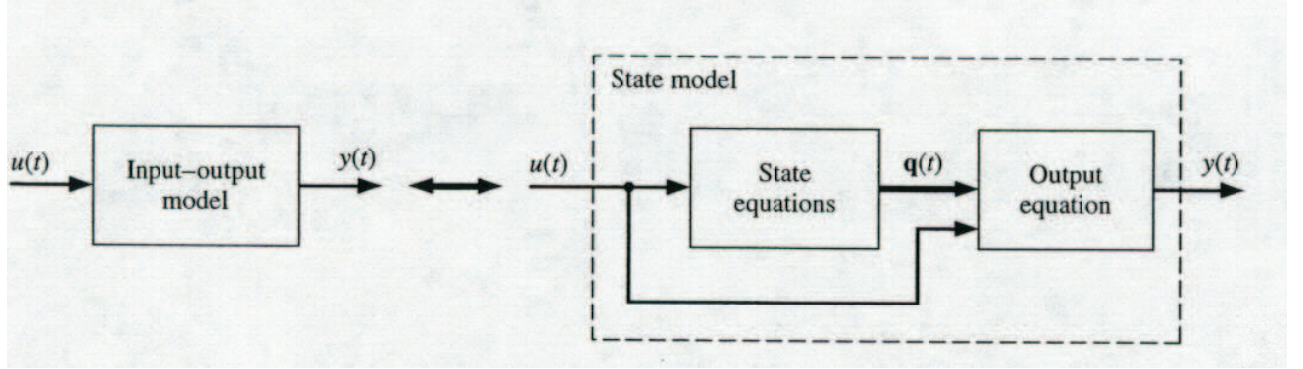


Figure 6: Equivalent input-output and state-space models.

Consider the following simple input-output equation

$$a_n \frac{d^n y(t)}{dt^n} + \dots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_0 u(t), \quad (19)$$

If we select the following state variables

$$x_1(t) = y(t), x_2(t) = \frac{dy(t)}{dt}, \dots, x_n(t) = \frac{d^{n-1}y(t)}{dt^{n-1}}, \quad (20)$$

the equivalent set of state-space equations are

$$\dot{x}_1(t) = x_2(t), \quad (21)$$

$$\dot{x}_2(t) = x_3(t) \quad (22)$$

$$\vdots \quad (23)$$

$$\dot{x}_{n-1}(t) = x_n(t) \quad (24)$$

$$\dot{x}_n(t) = -\left(\frac{a_0}{a_n}\right)x_1(t) - \left(\frac{a_1}{a_n}\right)x_2(t) - \dots - \left(\frac{a_{n-1}}{a_n}\right)x_n(t) + \left(\frac{b_0}{a_n}\right)u(t). \quad (25)$$

Figure 6 depicts the equivalence between input-output and state-space models.

Example 3

The input-output equation for a mechanical system is given by

$$(m_1 m_2) \frac{d^4 x(t)}{dt^4} + (m_2 b_1 + m_2 b_2 + m_1 b_2) \frac{d^3 x(t)}{dt^3}$$

$$\begin{aligned}
& + (m_1 k_1 + m_2 k_1 + m_2 k_2 + b_1 b_2) \frac{d^2 x(t)}{dt^2} \\
& + (b_1 k_2 + b_2 k_1) \frac{dx(t)}{dt} \\
& + k_1 k_2 x(t) = k_2 F(t),
\end{aligned} \tag{26}$$

where the input is $F(t)$ and the output is $x(t)$. Derive an equivalent state-space model for this system.

Define the state variables as

$$q_1(t) = x(t), q_2(t) = \frac{dx(t)}{dt}, q_3(t) = \frac{d^2 x(t)}{dt^2}, q_4(t) = \frac{d^3 x(t)}{dt^3}, \tag{27}$$

and the state-space equations are

$$\dot{q}_1(t) = q_2(t), \tag{28}$$

$$\dot{q}_2(t) = q_3(t), \tag{29}$$

$$\dot{q}_3(t) = q_4(t), \tag{30}$$

$$\begin{aligned}
\dot{q}_4(t) = & -\left(\frac{k_1 k_2}{m_1 m_2}\right)q_1(t) - \left(\frac{b_1 k_2 + b_2 k_1}{m_1 m_2}\right)q_2(t) - \left(\frac{m_1 k_2 + m_2 k_1 + m_2 k_2 + b_1 b_2}{m_1 m_2}\right)q_3(t) \\
& - \left(\frac{m_2 b_1 + m_2 b_1 + m_1 b_2}{m_1 m_2}\right)q_4(t) + \left(\frac{k_2}{m_1 m_2}\right)F(t).
\end{aligned} \tag{31}$$

The output equation is

$$y(t) = q_1(t). \tag{32}$$

Equations (28) through (32) form a state-space model of this mechanical system.

Let us now consider the transformation of a model to state-space form from the standard second-order form of mechanical systems. Consider the following mechanical system model obtained following repeated application of Newton's second law

$$\mathbf{m}\{\ddot{x}(t)\} + \mathbf{b}\{\dot{x}(t)\} + \mathbf{k}\{x(t)\} = \{F(t)\}, \tag{33}$$

where $\{\ddot{x}(t)\}$, $\{\dot{x}(t)\}$, $\{x(t)\}$ represent acceleration, velocity, and displacement vectors, and where $\mathbf{m}$, $\mathbf{b}$, $\mathbf{k}$ represent the mass, damping and stiffness

matrices. The vector $\{F(t)\}$ represents the forcing function of the system. Equation (33) can be rewritten as

$$\{\ddot{x}(t)\} + \mathbf{m}^{-1}\mathbf{b}\{\dot{x}(t)\} + \mathbf{m}^{-1}\mathbf{k}\{x(t)\} = \mathbf{m}^{-1}\{F(t)\}. \quad (34)$$

Now define the following state vector

$$\mathbf{q}(t) = \begin{bmatrix} \{x(t)\} \\ \{\dot{x}(t)\} \end{bmatrix}, \quad (35)$$

and the following input vector

$$\mathbf{u}(t) = \{F(t)\}. \quad (36)$$

Using the state vector (35) the second-order system can be written as

$$\dot{\mathbf{q}}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{m}^{-1}\mathbf{k} & -\mathbf{m}^{-1}\mathbf{b} \end{bmatrix} \mathbf{q}(t) + \begin{bmatrix} \mathbf{0} \\ \mathbf{m}^{-1} \end{bmatrix} \mathbf{u}(t). \quad (37)$$

What about the system outputs? As mentioned before, one can arbitrarily define any (linear) combination of states as the outputs for this system. For example, let us define all of the velocities as the outputs of this system, i.e. define the output vector $\mathbf{y}(t)$ as the vector $\{\dot{x}(t)\}$. Then we can express it in terms of the state vector as follows:

$$\mathbf{y}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{I} \end{bmatrix} \mathbf{q}(t) + \begin{bmatrix} \mathbf{0} \end{bmatrix} \mathbf{u}(t). \quad (38)$$

Equations (37) and (38) form a state-space representation of the forced mechanical system (34). This representation is appropriate for any M degree-of-freedom (MDOF) mechanical system. Furthermore, the mechanical system (34) contains only translational dynamics, i.e. only linear displacement, velocity and acceleration are involved in the equations. A similar process of deriving state-space equations applies to mechanical systems that include only rotational dynamics and even both translational and rotational dynamics.

The definition of the state vector must be augmented to include the rotational degrees of freedom of the system, that is the angular displacements and angular velocities.

Reading Assignment

See separate file on textbook reading assignments depending on the text edition you own. Read Handout A.3 and examples Handout E.7 posted on the course web page.