Texas A & M University
Department of Mechanical Engineering
MEEN 364 Dynamic Systems and Controls
Dr. Alexander G. Parlos

Lecture 21: Steady-State System Tracking and System Stability

The objective of this lecture is to discuss certain aspects of steady-state feedback system tracking issues, and look into the methods by which we can determine system stability given a transfer function or a state-space model.

Steady-State Tracking and System Type

In general, we would like to investigate the steady-state performance of a feedback system to disturbances and/or reference inputs that are not only constants, but can be expressed as arbitrary polynomials in time. Assuming that the system under consideration is **stable**[1], we classify such systems into **system types** depending to the polynomial degree for which the tracking error is constant. System types can be defined with respect to the reference input or disturbance.

Let us consider the system with block diagram representation given by Figure 1. This block diagram can be simplified as shown in Figure 2, where we have ignored the disturbance signal. So, we will first concentrate on the steady-state tracking performance of the system to reference inputs, $r(t)$.

Now, let's consider the following reference input

$$r(t) = \frac{t^k}{k!}1(t),\tag{1}$$

with the corresponding Laplace transform

$$R(s) = \frac{1}{s^{k+1}}.\tag{2}$$

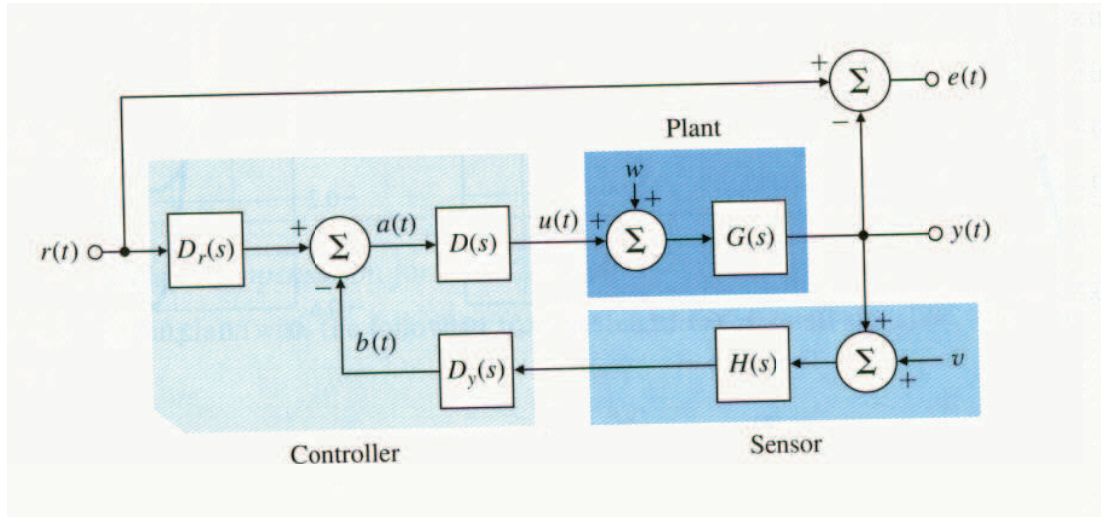[1]Otherwise, steady-state response does not even exist.

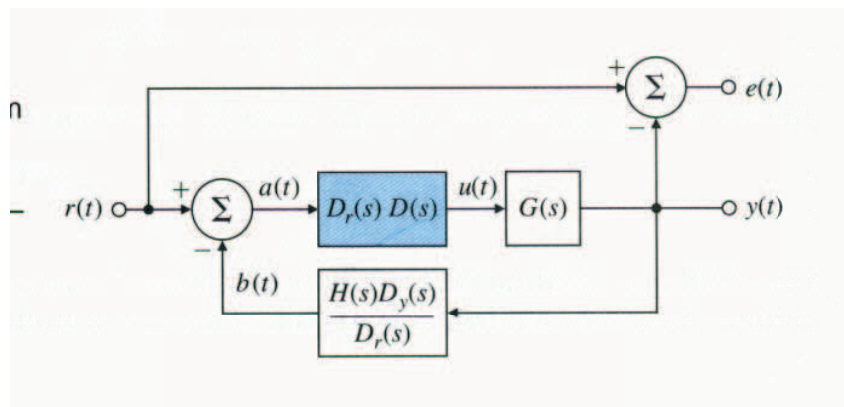Figure 1: Typical single-loop system.

Figure 2: Simplified block diagram of a typical single-loop system.

To compute the steady-state errors we start from the following reference to output transfer function (see Figure 2)

$$\frac{Y(s)}{R(s)} = T(s) = \frac{D_r DG(s)}{1 + HD_y DG(s)},$$
(3)

whereas the error is expressed as

$$E(s) = R(s) - Y(s) = R(s) - T(s)R(s),$$
(4)

or

$$E(s) = (1 - T(s))R(s).$$
(5)

From the final value theorem

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s(1 - T(s))R(s).$$
(6)

So, the steady-state error can be expressed as

$$e_{ss} = \lim_{s \to 0} s\frac{1 - T(s)}{s^{k+1}} = \lim_{s \to 0} \frac{1 - T(s)}{s^k}.$$
(7)

The result of the above expression can be zero, a non-zero constant, or infinity. If it is a non-zero constant, the system is referred to as **Type k**.

The Special Case of Unity Feedback

Assume now that the system has unity feedback, with $D_r(s) = 1$, and that

$$G_0(s) = D_r DG(s).$$
(8)

Then, from the closed-loop transfer function we obtain that

$$1 - T(s) = \frac{1}{1 + G_0(s)}.$$
(9)

The system error is given by

$$E(s) = \frac{1}{1 + G_0(s)}R(s).$$
(10)

Using the Final Value Theorem, we find that the steady-state value of the system error is

$$e_{ss} = \lim_{s \to 0} s\frac{1}{(1 + G_0(s))s^{k+1}} = \lim_{s \to 0} \frac{1}{(1 + G_0(s))s^k}.$$
(11)

	Step	Ramp	Parabola
Type 0	$\frac{1}{1+K_p}$	∞	∞
Type I	0	$\frac{1}{K_v}$	∞
Type II	0	0	$\frac{1}{K_a}$

Now, depending on the system type and the nature of the reference signal, we have differing steady-state errors. These are summarized in the following table.

The various coefficients mentioned in the table are defined as

$$Position - error\ Constant\ K_p = \lim_{s \to 0} G_0(s), \qquad (12)$$

$$Velocity\ Constant\ K_v = \lim_{s \to 0} sG_0(s), \qquad (13)$$

$$Acceleration\ Constant\ K_a = \lim_{s \to 0} s^2 G_0(s). \qquad (14)$$

System Type with Respect to Disturbance

Similar to the ability of a system to track references, it is possible to define system type with respect to disturbance rejection. To do so, we define the following transfer function

$$\frac{Y(s)}{W(s)} = T_w(s). \qquad (15)$$

This transfer function determines the system error from a disturbance input.

The system is *Type 0* if a step disturbance results in a constant steady-state output error. The system is *Type 1* if, for a ramp input disturbance, the steady-state value of the output is a constant, i.e.

$$y_{ss} = \lim_{s \to 0}[sT_w(s)\frac{1}{s^2}] = constant. \qquad (16)$$

or equivalently, the steady-state output for a step input disturbance is,

$$y_{ss} = \lim_{s \to 0}[sT_w(s)\frac{1}{s}] = 0. \qquad (17)$$

In other words,

$$T_w(0) = 0, \qquad (18)$$

or DC gain of zero.

System Stability

There are varying definitions for stability, but we will choose to use the concept of **bounded input-bounded output (BIBO) stability**.

That is a system is said to be BIBO stable, if for every bounded input, $u(t)$, the resulting system response, $y(t)$, is also bounded. In terms of the system impulse response, $h(t)$, BIBO stability can be stated as follows: *a system is BIBO stable if and only if the integral*

$$\int_{-\infty}^{-\infty} |h(\tau)|d\tau < \infty. \tag{19}$$

In general, if a system has any poles on the imaginary axis or on the RHP, the integral of its impulse response will not be finite and it will not be a BIBO-stable system. Whereas, if all of its poles are in the LHP, then it will be BIBO-stable. So, we can use the pole locations of a system to determine stability.

Equivalence of Transfer Function Poles and System Eigenvalues: In the past, we derived transfer function models for dynamic systems and defined the poles of such transfer functions as the zeroes of the denominator polynomial. Further, we have developed state-space models, either directly from physical principles or from transfer function models. We then defined the system modes, as the eigenvalues of the **A** matrix of the state-space model. For a given system transfer function model and its equivalent state-space representation, **the poles of the system are identical to the eigenvalues of the A matrix of the system state-space representation**. As a result, we can investigate system stability using system poles (in the frequency domain) or system eigenvalues (in the time domain).

Stability Criterion Based on Poles or Eigenvalues

Consider the transfer function

$$T(s) = \frac{Y(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_m}{s^n + a_1 s^{n-1} + \ldots + a_n},$$
$$= \frac{K \prod_{i=1}^{m}(s - z_i)}{\prod_{i=1}^{n}(s - p_i)}, \quad m \leq n, \tag{20}$$

where p_i are the system poles and z_i are the system zeros. The denominator polynomial set to zero is also called the *characteristic equation.*

The system response in the time-domain can now be expressed as

$$y(t) = \sum_{i=1}^{n} K_i e^{p_i t}, \tag{21}$$

where the exponents are the poles and the coefficients depend on the system zeroes and the initial conditions. This system will be stable if and only if all terms on the right-hand-side of equation (21) go to zero as $t \rightarrow \infty$. This happens if all of the poles of the system are strictly on the LHP, i.e.

$$\Re e \{p_i\} < 0. \tag{22}$$

This is called **asymptotic internal stability**. Since the poles of a system are identical to the eigenvalues of its equivalent state-space representation, we can also state that a system will be stable if all of its eigenvalues are strictly on the LHP, i.e.

$$\Re e \{\lambda_i\} < 0. \tag{23}$$

So, we can determine the stability of a system by computing its poles or its eigenvalues and checking their real-parts. Computing the poles of a system from a general order denominator polynomial or computing the eigenvalues of a system from a general matrix requires tools such as MATLAB.

Routh's Stability Criterion

A more indirect way of determining system stability without the use of MATLAB is the so-called **Routh criterion**. This criterion states that *a necessary and sufficient condition for stability is that all of the elements in the first column of the Routh array be positive.*

Row n	s^n :	1	a_2	a_4	$\cdots$
Row n-1	s^{n-1}:	a_1	a_3	a_5	$\cdots$
Row n-2	s^{n-2}:	b_1	b_2	b_3	$\cdots$
Row n-3	s^{n-3}:	c_1	c_2	c_3	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
Row 2	s^2:	*	*		
Row 1	s^1:	*			
Row 0	s^0:	*			

To construct the Routh array, we consider the following characteristic equation

$$a(s) = s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n = 0. \tag{24}$$

Then the Routh array is constructed as shown in the table above.

The elements b and c in the array are computed as follows:

$$
\begin{aligned}
b_1 &= -\frac{1a_3 - a_1 a_2}{a_1} = \frac{a_1 a_2 - a_3}{a_1}, \\
b_2 &= -\frac{1a_5 - a_1 a_4}{a_1} = \frac{a_1 a_4 - a_5}{a_1}, \\
b_3 &= -\frac{1a_7 - a_1 a_6}{a_1} = \frac{a_1 a_6 - a_7}{a_1}, \\
c_1 &= -\frac{a_1 b_2 - b_1 a_3}{b_1} = \frac{b_1 a_3 - a_1 b_2}{b_1}, \\
c_2 &= -\frac{a_1 b_3 - b_1 a_5}{b_1} = \frac{b_1 a_5 - a_1 b_3}{b_1}, \\
c_3 &= -\frac{a_1 b_4 - b_1 a_7}{b_1} = \frac{b_1 a_7 - a_1 b_4}{b_1}.
\end{aligned}
\tag{25}
$$

Example: Routh's Test

Determine the stability of a system that has the following characteristic polynomial

$$a(s) = s^6 + 4s^5 + 3s^4 + 2s^3 + s^2 + 4s + 4 = 0. \tag{26}$$

The Routh array is given in the table above.

s^6:	1	3	1	4
s^5:	4	2	4	0
s^4:	$\frac{5}{2} = \frac{4\cdot 3 - 1\cdot 2}{4}$	$0 = \frac{4\cdot 1 - 4\cdot 1}{4}$	$4 = \frac{4\cdot 4 - 1\cdot 0}{4}$	
s^3:	$2 = \frac{\frac{5}{2}\cdot 2 - 4\cdot 0}{\frac{5}{2}}$	$-\frac{12}{5} = \frac{\frac{5}{2}\cdot 4 - 4\cdot 4}{\frac{5}{2}}$	0	
s^2:	$3 = \frac{2\cdot 0 - (\frac{5}{2})\cdot(-\frac{12}{5})}{2}$	$4 = \frac{2\cdot 4 - \frac{5}{2}\cdot 0}{2}$		
s^1:	$-\frac{76}{15} = \frac{2\cdot(-\frac{12}{5}) - 8\cdot 1}{3}$	0		
s^0:	$4 = \frac{-\frac{76}{15}\cdot 4 - 0}{-\frac{76}{15}}.$			

Since the elements of the first column of the Routh array are not all positive, the polynomial has RHP roots. In fact, there are two roots on the RHP, since there are two sign changes on the first column of the Routh array.

Reading Assignment

See separate file on textbook reading assignments depending on the text edition you own. Read the examples in Handout E.22 and Handout A.6 posted on the course web page.