Texas A & M University
Department of Mechanical Engineering
MEEN 364 Dynamic Systems and Controls
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Lecture 22: Introduction to Root-Locus

The objective of this lecture is to introduce you to another method used in the analysis and design of control systems, namely the root-locus technique. The basic idea exploited in root-locus design is that one can track the movement of the closed-loop poles as parameters of interests, such as a controller gain, are varied. This gives a designer the map of the closed-loop pole movement on the s-plane and proper design choices can be made.

Root Locus of a Basic Feedback System

Assume the following closed-loop transfer function

$$\frac{Y(s)}{R(s)} = \frac{K_A K_G G(s)}{1 + K_A K_G G(s)}, \tag{1}$$

where K_A and K_G is the controller and system gain, respectively. The closed-loop poles are the roots of

$$1 + K_A K_G G(s) = 0. \tag{2}$$

Let us assume that the transfer function $G(s)$ can be expressed in terms of its poles and zeros as

$$G(s) = \frac{b(s)}{a(s)} = \frac{\Pi_{i=1}^{m}(s - z_i)}{\Pi_{i=1}^{n}(s - p_i)}, \tag{3}$$

and that

$$K = K_A K_G. \tag{4}$$

We now express the characteristic equation in various root-locus forms, as follows:

$$1 + KG(s) \;=\; 0, \tag{5}$$

$$\begin{aligned} 1 + K\frac{b(s)}{a(s)} &= 0, \qquad (6) \\ a(s) + Kb(s) &= 0, \qquad (7) \\ G(s) &= -\frac{1}{K}. \qquad (8) \end{aligned}$$

The root-locus can be thought of as a method for inferring the location of the closed-loop poles from examining the open-loop transfer function, $KG(s)$.

Example 1: Root Locus with Respect to Controller Gain

A normalized transfer function of a DC motor is

$$\frac{\theta_m(s)}{V_a(s)} = K_G G(s) = \frac{1}{s(s+1)}. \qquad (9)$$

The characteristic equation for this system is

$$1 + K\frac{1}{s(s+1)} = 0, \qquad (10)$$

or

$$s^2 + s + K = 0. \qquad (11)$$

The solution of equation (11) gives the closed-loop pole locations, as follows:

$$r_1, r_2 = -\frac{1}{2} \pm \frac{\sqrt{1-4K}}{2}. \qquad (12)$$

Plotting these two roots as K is allowed to vary, results in the following root-locus, shown in Figure 1.

Several simple observations can be made from Figure 1. There are two roots and thus two branches to the locus. At $K = 0$ these branches begin at the open-loop poles, because for $K = 0$ the system is open-loop. As K is increased the closed-loop poles move towards each other and they meet at $s = -\frac{1}{2}$. At that point they break away from the real axis. From the **breakaway point** the poles move towards infinity, while their sum is equal to -1. We have now characterized the path the closed-loop poles transverse as the gain K is allowed to increase.

Example 2: Root Locus with Respect to Plant Parameters

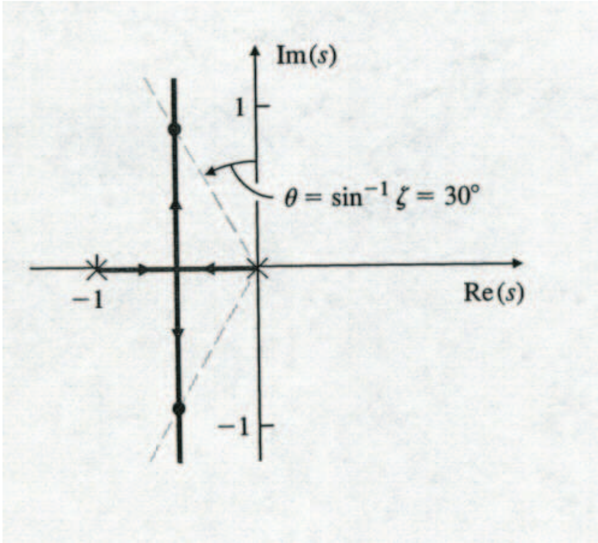

Figure 1: Root Locus for G(s) $= \frac{1}{s(s+1)}$ as a function of the controller gain K.

Now, consider the following transfer function

$$G(s) = \frac{1}{s(s+c)}. \tag{13}$$

We want to find the root-locus with respect to c. Here we assume that $K = 1$. The corresponding closed-loop characteristic equation is

$$1 + G(s) = 0, \tag{14}$$

or

$$s^2 + cs + 1 = 0. \tag{15}$$

We can express this as

$$1 + c\frac{s}{s^2+1} = 0, \tag{16}$$

placing it in the standard root-locus form. The roots of equation (16) can be expressed as

$$r_1, r_2 = -\frac{c}{2} \pm \frac{\sqrt{c^2-4}}{2}. \tag{17}$$

Plotting these two roots as c is allowed to vary results in the following root-locus, shown in Figure 2.

Note that when $c = 0$ the closed-loop poles are also the open-loop poles. The closed-loop poles are damped as c grows. At $s = -1$ the two segments

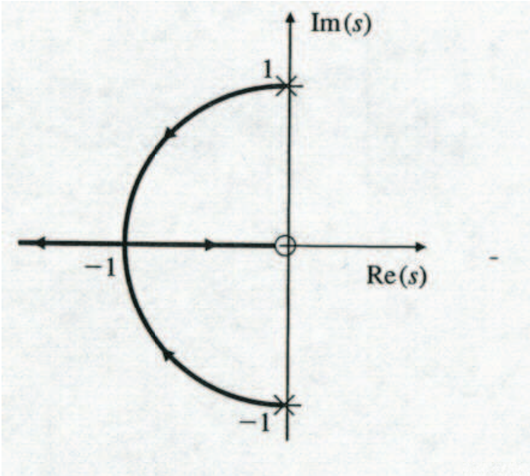

Figure 2: Root Locus for G(s) = $\frac{1}{s(s+c)}$ as a function of the damping factor c.

of the root-locus abruptly change direction and move away from each other; one towards the origin and the other towards infinity. This point of change in direction is called the **break-in point**.

Guidelines for Sketching a Root Locus

The root-locus of a transfer function can be accurately drawn using, for example, MATLAB. However, one can obtain sketches of the root-locus by following the following steps. We will demonstrate the use of these steps on the following transfer function

$$G(s) = \frac{1}{s((s+4)^2 + 16)}. \tag{18}$$

STEP 1: Place the open-loop poles and zeros of $G(s)$ on the s-plane. See Figure 3.

STEP 2: Find the segments of the real-axis belonging to the root-locus. The acceptable segments must have an odd number of poles plus zeros to their right. See Figure 4.

STEP 3: Draw the asymptotes for large values of K. As $K \rightarrow \infty$ the

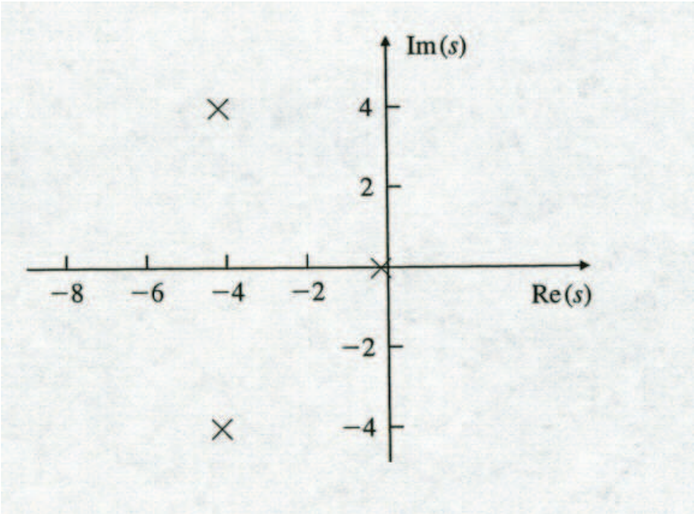

Figure 3: Step 1 of the Root Locus.

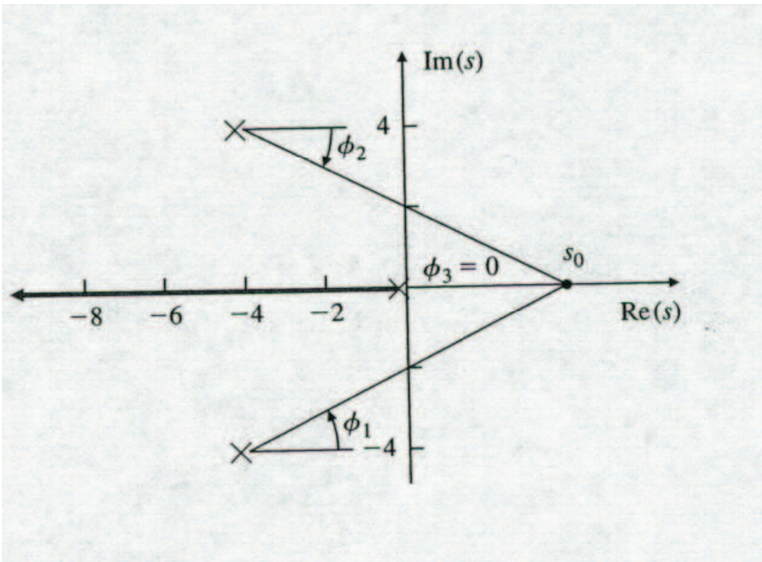

Figure 4: Step 2 of the Root Locus.

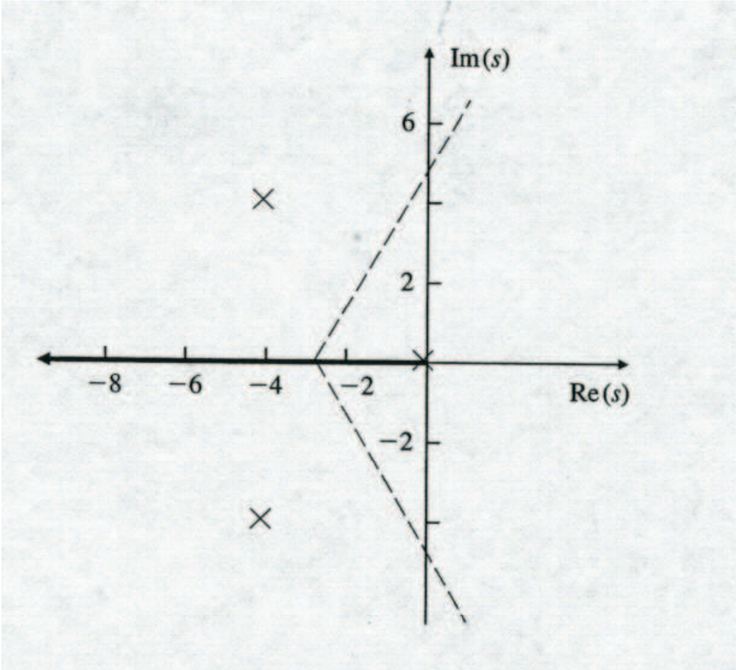

Figure 5: Step 3 of the Root Locus.

root-locus equation

$$G(s) = -\frac{1}{K}, \tag{19}$$

can only be satisfied if $G(s) = 0$. This can occur in two ways; first at the zeros of $G(s)$ and second when some asymptotes approach infinity. If the system has n poles and m zeros, then there will be $n - m$ asymptotes. These asymptotes form angles (with the real axis) given by

$$\phi_l = \frac{180^o + 360^o(l-1)}{n-m}, \; l = 1, 2, \ldots, n-m, \tag{20}$$

and they intercept the real-axis at

$$\alpha = \frac{\sum p_i - \sum z_i}{n-m}. \tag{21}$$

See Figure 5.

STEP 4: Compute the departure angles from the open-loop poles and the arrival angles to the open-loop zeros. If a pole appears q times then the departure angle is given by

$$q\phi_{dep} = \sum \psi_i - \sum \phi_i - 180^o - 360^o l, \tag{22}$$

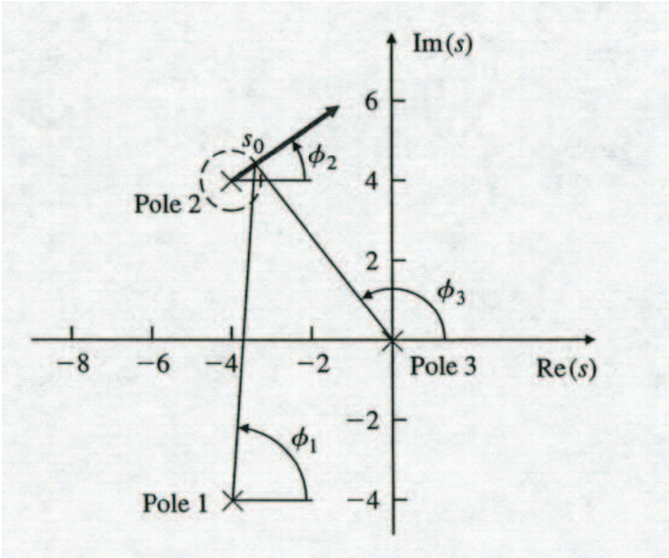

Figure 6: Step 4 of the Root Locus.

where $\sum \phi_i$ is the sum of the angles of the remaining poles and $\sum \psi_i$ is the sum of the angles of all the zeros. Also, l takes q values and it is selected such that ϕ_{dep} in the range $(-180^o, +180^o)$. Similarly, for arrival angles, we have the following formula

$$q\psi_{arr} = \sum \phi_i - \sum \psi_i + 180^o + 360^o l, \tag{23}$$

See Figure 6.

STEP 5: Estimate the points where the root-locus crosses the imaginary axis. This can be done using Routh's criterion. For the third-order example we are using, the characteristic equation is

$$1 + \frac{K}{s((s+4)^2+16)} = 0, \tag{24}$$

or equivalently,

$$s^3 + 8s^2 + 32s + K = 0. \tag{25}$$

The Routh array is then given by

$$\begin{array}{lccc} s^3 & : & 1 & 32 \\ s^2 & : & 8 & K \end{array}$$

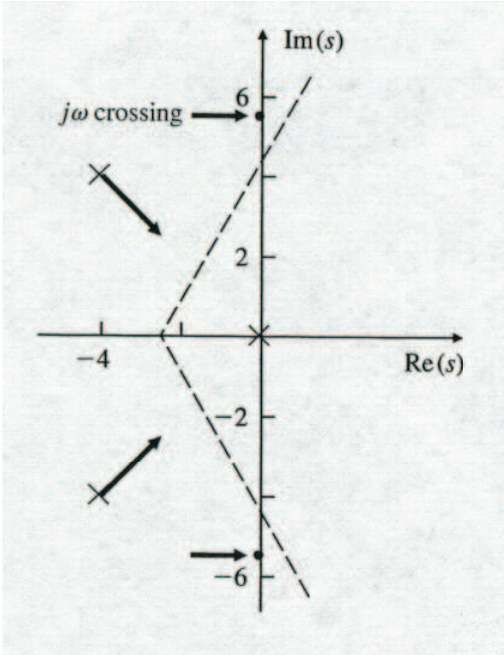

Figure 7: Step 5 of the Root Locus.

$$
\begin{array}{lll}
s^1 : & \dfrac{8 \times 32 - K}{8} & 0 \\
s^0 : & K &
\end{array}
\tag{26}
$$

For $0 < K < 256$ there are no positive roots of the characteristic polynomial. If we substitute $K = 256$ and $s = j\omega$ into equation (25) we obtain the non-trival solution $\omega = \pm 5.66$. Note that the asymptote crosses the imaginary axis at 4.62. See Figure 7.

STEP 6: Estimate the location of multiple roots, especially on the real-axis (breakaway points). The location s_0 of the breakaway point is computed by solving the following equation

$$\frac{d}{ds}\left(-\frac{1}{G(s)}\right)_{s=s_0} = 0. \tag{27}$$

STEP 7: Complete the root-locus sketch by combining the information obtained from Steps 1 through 5, and 6 if necessary. See Figure 8.

Uses and Gain Selection from the Root Locus

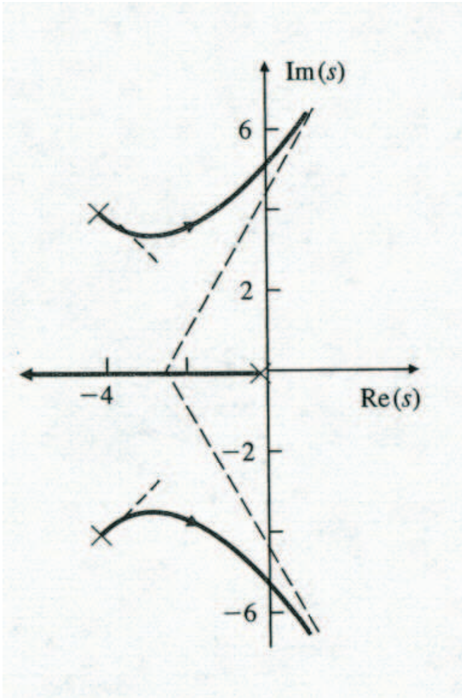

Figure 8: Step 7 of the Root Locus.

The root-locus technique is a method by which the closed-loop pole locations of a system can be tracked as a parameter is allowed to vary from zero to infinity. So far we only mentioned the use of the control gain as a parameter to vary. However, any system parameter can be allowed to vary and the same procedures of root-locus can be used in mapping the location of the closed-loop poles. The key is to express the characteristic equation in the form

$$1 + KG(s) = 0, \tag{28}$$

where K can be any parameter to be varied. In doing so, one would collect the terms that do not multiply the parameter to be varied and name it $a(s)$, whereas the terms that do multiple the parameter as $b(s)$. These two polynomials define the poles and zeros of the system, respectively. The rest of the procedure for plotting the root-locus is as described before. Further, one could generate a root-locus when two parameters must be allowed to vary. This is done by allowing one parameter to vary while fixing the other.

One more use of the root-locus is that it allows us to compute the gain required to place the closed-loop poles on certain locations on the root-locus. Every point on the root-locus satisfies the condition

$$1 + KG(s) = 0. \tag{29}$$

This is a complex relation and the magnitude part of equation (29) corresponds to

$$K = \frac{1}{|G(s)|}. \tag{30}$$

So, for any point on the root-locus, s_0, the required gain K_0 can be obtained from

$$K_0 = \frac{1}{|G(s)|_{s=s_0}}. \tag{31}$$

Reading Assignment

See separate file on textbook reading assignments depending on the text edition you own. Read the examples in Handout E.26 posted on the course web page.