Texas A & M University
Department of Mechanical Engineering
MEEN 364 Dynamic Systems and Controls
Dr. Alexander G. Parlos

Lecture 9: Linearization and Scaling – Operating Points and Impedance Matching

The objective of this lecture is to give you an overview of the mathematical method involved in linearizing the dynamics of nonlinear systems, in order to express them in standard state-space form. Linearization allows us to analyze complex dynamics using simple mathematics and analytical methods rather than computer simulations.

Linearization

The differential equations of motion for most practically interesting systems are nonlinear. For example, most useful forms of damping contains nonlinear terms. As mentioned before, it is much easier to deal with linear models of a system than nonlinear ones.

Linearization is the process of finding a linear model of a system that approximates a nonlinear one. Over 100 years ago, Lyapunov proved that if a linearized model of a system is valid near an equilibrium point of the system and if this linearized model is stable, then there is a region around this equilibrium point that contains the equilibrium, within which the nonlinear system is also stable. Basically this tells us that, at least within a region of an equilibrium point, we can investigate the behavior of a nonlinear system by analyzing the behavior of a linearized model of that system. This form of linearization is also called *small-signal linearization.*

Assume that the nonlinear equations of motion for a system model with one controlled input, u, are expressed in the form

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, u). \tag{1}$$

In equation (1) the derivatives of the state are relate to the state and/or the control through a nonlinear relation $\mathbf{f}$. In order to linearize this equation we must first determine the equilibrium values of the system. The equilibrium values for the state, $\mathbf{x}_0$, and control, u_0 are such that the derivative of the state vector is zero. That is, we can compute the equilibrium values by solving

$$\dot{\mathbf{x}}_0 = 0 = \mathbf{f}(\mathbf{x}_0, u_0). \tag{2}$$

Equation (2) has two unknowns. Therefore, we must choose arbitrarily the value of u_0 and solve equation (2) for the equilibrium state, $\mathbf{x}_0$.

We now expand the nonlinear equation in terms of the perturbations from these equilibrium values; that is, we let

$$\mathbf{x}(t) = \mathbf{x}_0 + \delta\mathbf{x}(t), \tag{3}$$

and

$$u(t) = u_0 + \delta u(t), \tag{4}$$

then we can write the following linear approximation to the nonlinear dynamics (1)

$$\dot{\mathbf{x}}_0 + \delta\dot{\mathbf{x}}(t) \approx \mathbf{f}(\mathbf{x}_0, u_0) + \mathbf{F}\delta\mathbf{x}(t) + \mathbf{G}\delta u(t), \tag{5}$$

where $\mathbf{F}$ and $\mathbf{G}$ are the best linear fits to the nonlinear function $\mathbf{f}$ at the point $(\mathbf{x}_0, u_0)$. Canceling the equilibrium from both sides of equation (5) results in

$$\delta\dot{\mathbf{x}}(t) = \mathbf{F}\delta\mathbf{x}(t) + \mathbf{G}\delta u(t), \tag{6}$$

which is a linear model approximating the nonlinear dynamics at the point $(\mathbf{x}_0, u_0)$.

If an analytical expression for $\mathbf{f}$ is available, then the best linear fits $\mathbf{F}$ and $\mathbf{G}$ can be obtained through differentiation, as follows:

$$\mathbf{F} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}|_{(\mathbf{x}_0, u_0)}, \tag{7}$$

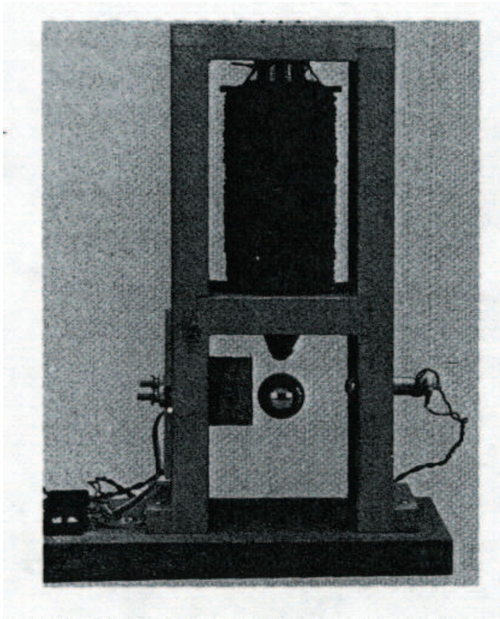

Figure 1: Laboratory scale magnetic ball levitator.

and

$$\mathbf{G} = \frac{\partial \mathbf{f}}{\partial u}\Big|_{(\mathbf{x}_0, u_0)}. \tag{8}$$

In case when an analytical expression is not available, numerical differentiation is performed, as shown in the following example.

Note:

For more details regarding linearization of nonlinear dynamics read the handout A.5.

Example: Linearization of Motion in a Ball Levitator

Figure 1 shows a laboratory scale magnetic levitator, where one electromagnet is used to levitate a ball bearing. The physical arrangement of the levitator is depicted in Figure 2.

The equation of motion for the ball is derived from Newton's law as

$$m\ddot{x}(t) = f_m(x, i) - mg, \tag{9}$$

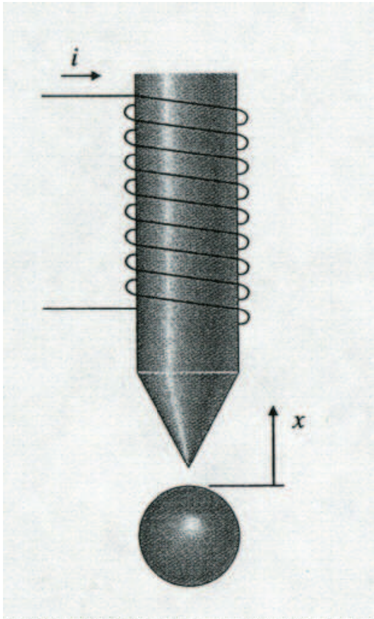

Figure 2: Model for a ball levitator.

where the force $f_m(x, i)$ is caused by the field of the electromagnet. Theoretically speaking, the force is proportional to the inverse of the square distance from the magnet, but the exact expression is difficult to derive. So, we do not have an analytic expression for the force. However, the force can be measured and plotted. Figure 3 shows the experimental curves for a ball with a 1 cm diameter mass and a mass of 8.4×10^{-3} kg.

From the experimental curves we infer that at the current value of $i_2 = 600$ mA and the displacement x_1, the magnetic force f_m just cancels the gravity force $mg = 82 \times 10^{-3}$ N. The mass is 8.4×10^{-3} kg and the acceleration of gravity is $9.8 \frac{m}{sec^2}$. Therefore, the point (x_1, i_2) represents an equilibrium point.

We now want to find the linearized equations of motion for this system. First, we expand the magnetic force in terms of deviations from the equilibrium point (x_1, i_2), as follows:

$$f_m(x_1 + \delta x, i_2 + \delta i) \approx f_m(x_1, i_2) + K_x \delta x + K_i \delta i. \tag{10}$$

The linear gains K_x and K_i can be computed as follows. K_x is the slope (or

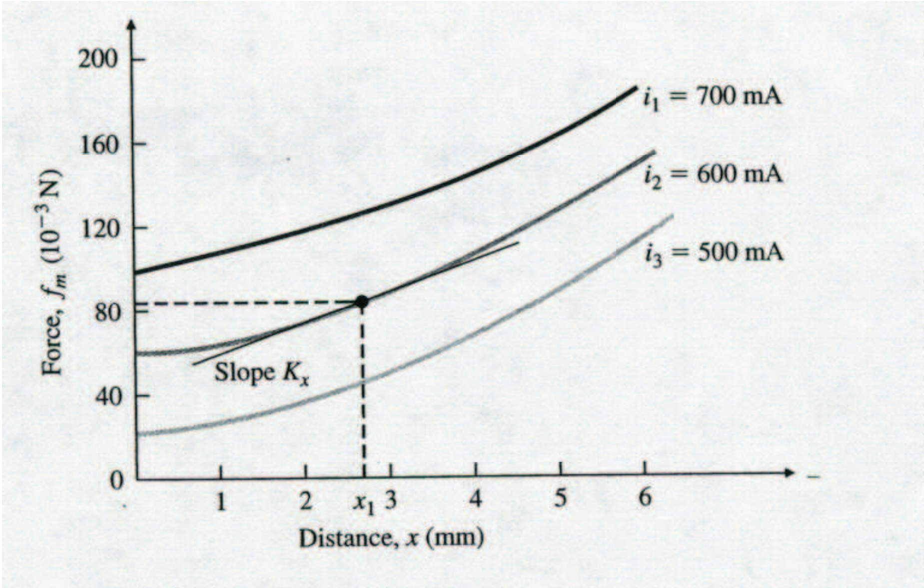

Figure 3: Experimentally determined force curves.

derivative) of the curve in Figure 3 for $i_2 = 600\ mA$ around the point x_1. This is found to be about 14 $\frac{N}{m}$. K_i is the change of force with current at the value $x = x_1$. This is found as

$$K_i \approx \frac{122 \times 10^{-3} - 42 \times 10^{-3}}{700 - 500} \approx 0.4\ \frac{N}{A}. \tag{11}$$

So, the linearized force expression becomes

$$f_m(\delta x, \delta i) = 82 \times 10^{-3} + 14\delta x + 0.4\delta i. \tag{12}$$

Considering that $\ddot{x}(t) = \delta\ddot{x}(t)$, the equation of motion (9) becomes

$$8.4 \times 10^{-3}\delta\ddot{x}(t) = 14\delta x(t) + 0.4\delta i(t), \tag{13}$$

or

$$\delta\ddot{x}(t) = 1667\delta x(t) + 47.6\delta i(t), \tag{14}$$

which is the linearized equations of motion about the equilibrium point. We can select the state vector as $\mathbf{x}(t) = [\delta x(t), \delta\dot{x}(t)]$ and the control $u(t) = \delta u(t)$. This selection results in the following state matrices

$$\mathbf{F} = \begin{bmatrix} 0 & 1 \\ 1667 & 0 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 0 \\ 47.6 \end{bmatrix}. \tag{15}$$

Amplitude Scaling

Amplitude scaling is usually performed by simply picking units that make sense for the problem investigated. In selecting the units we try to make the numbers of the problem comparable. For example, for the ball levitator expressing the displacement in millimeters and the current in milliamps would make the numbers easy to work with. A method for accomplishing the best scaling for a complex system is first to estimate the maximum values for each state and then scale the system so that each element varies between -1 and $+1$.

The amplitude scaling is performed by defining the scaled variables for each element: If

$$x'(t) = S_x x(t), \tag{16}$$

then

$$\dot{x}'(t) = S_x \dot{x}(t); \quad \ddot{x}'(t) = S_x \ddot{x}(t). \tag{17}$$

We select the scaling S_x to accomplish our scaling objective outline above.

Time Scaling

Variables involving time are usually measured in units of seconds. Sometimes it is convenient to express time in other units. We define a scaled time to be

$$\tau = \omega_0 t, \tag{18}$$

such that if t is measured in seconds and $\omega_0 = 1000$ then τ will be measured in milliseconds. The effect of time scaling is to change the differentiation as follows

$$\dot{x} = \frac{dx}{dt} = \frac{dx}{d(\tau/\omega_0)} = \omega_0 \frac{dx}{d\tau}, \tag{19}$$

and

$$\ddot{x} = \frac{d^2x}{dt^2} = \omega_0^2 \frac{d^2x}{d\tau^2}. \tag{20}$$

If the original system is in state-variable (or state space) form

$$\dot{\mathbf{x}}(t) = \mathbf{F}\mathbf{x} + \mathbf{G}u, \tag{21}$$

then the time scaled system is expressed as

$$\dot{\mathbf{x}}(t) = \frac{1}{\omega_0}\mathbf{F}\mathbf{x} + \frac{1}{\omega_0}\mathbf{G}u. \tag{22}$$

Read example 2.25 on page 76 of the text.

Note:

For more on operating points read the handout A.5. For an exaple describing load (or impedance) matching read handout A.4.

Reading Assignment

See separate file on textbook reading assignments depending on the text edition you own. Read examples Handout E.12 and Handout A.5 posted on the course web page.