HANDOUT A.5 - LINEARIZATION OF NONLINEAR DYNAMICS

Introduction

The dynamics of a physical system can be expressed in the following general form

$$\dot{x}(t) = f\left(x(t), u(t)\right) \tag{1}$$

$$y(t) = h\left(x(t), u(t)\right) \tag{2}$$

where the functions, '*f*' and '*h*' are any nonlinear function of the variables 'x(t)' and 'u(t)'. It is very difficult to use a non-linear model for things other than simulations. Therefore an approximate model is obtained by linearizing the non-linear model about some operating point. To obtain this linearized model, the original variables are assumed to deviate only slightly about the operating point.

Finding the equilibrium point

To obtain the equilibrium point, all the derivative terms in the governing equation are equated to zero. In other words, in equation (1), $\dot{x}$ is equated to zero, i.e.,

$$\dot{x}(t) = 0$$

$$\Rightarrow f\left(x(t), u(t)\right) = 0. \tag{3}$$

For a given input, 'u (t)' by solving equation (3), the value of 'x(t)' can be obtained. This value of 'x(t)' for the corresponding value of the input, 'u(t)' is substituted in equation (2) to obtain a value of 'y(t)'. This set of 'x(t)', 'u(t)' and 'y(t)' is the operating point or the equilibrium point of the system. Assuming that the equilibrium point does not vary with time, we now have an equilibrium point consisting of the triplet $\left(u_0, x_0, y_0\right)$.

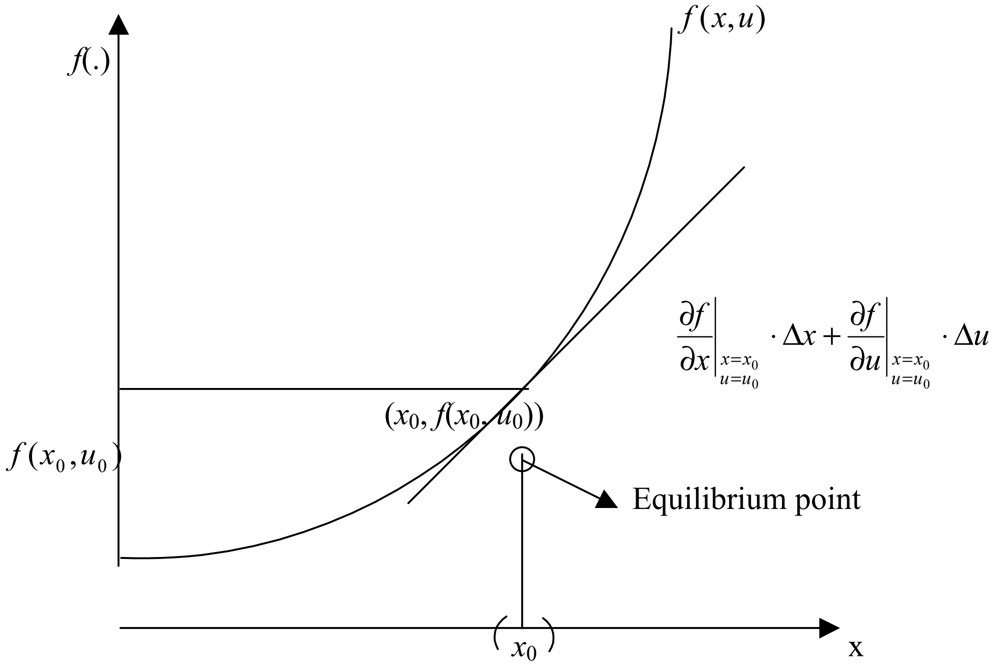
Taylor series expansion

From calculus it is known that if a function $f(x_1, x_2)$ is to be expanded using Taylor's series about the equilibrium point (x_{10}, x_{20}), then the Taylor series expansion of the function is given as

$$f(x_1, x_2) = f(x_{10}, x_{20}) + \left.\frac{\partial f}{\partial x_1}\right|_{\substack{x_1 = x_{10} \\ x_2 = x_{20}}} \cdot (x_1 - x_{10}) + \left.\frac{\partial f}{\partial x_2}\right|_{\substack{x_1 = x_{10} \\ x_2 = x_{20}}} \cdot (x_2 - x_{20}) + \left.\frac{\partial^2 f}{\partial x_1{}^2}\right|_{\substack{x_1 = x_{10} \\ x_2 = x_{20}}} \cdot (x_1 - x_{10})^2 +$$

$$\left.\frac{\partial^2 f}{\partial x_2{}^2}\right|_{\substack{x_1 = x_{10} \\ x_2 = x_{20}}} \cdot (x_2 - x_{20})^2 + \ldots$$

Representing the Linearization technique in the form of a figure, we get



From the above figure it can be noticed that, the curve *f(x, u)* is linearized about the equilibrium point $(u_0, f(x_0,u_0))$ and the result is a line tangent to the curve at the equilibrium point. The equation of the straight line is obtained by the Taylor series expansion. The linearity is valid only for a small interval around the equilibrium point as shown in the figure. The higher order terms in the expansion are neglected.

Once the operating point of the system is obtained, the non-linear function given by equation (1) is expanded using the Taylor series about the operating point, (u_0, x_0, y_0) as follows

$$
\begin{aligned}
\dot{x}(t) &= f(x(t),u(t)) \\
&= f(x_0,u_0) + \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_0\\u=u_0}} \cdot (x(t)-x_0) + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{\substack{x=x_0\\u=u_0}} \cdot (x(t)-x_0)^2 + \cdots + \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_0\\u=u_0}} \cdot (u(t)-u_0) \quad (4) \\
&+ \frac{1}{2!}\left.\frac{\partial^2 f}{\partial u^2}\right|_{\substack{x=x_0\\u=u_0}} \cdot (u(t)-u_0)^2 + \ldots
\end{aligned}
$$

where the derivatives $\frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial f}{\partial u}, \frac{\partial^2 f}{\partial u^2}$,... are evaluated at the equilibrium point $x = x_0$ and $u = u_0$.

Define

$$\Delta x(t) = x(t) - x_0, \qquad \Delta u(t) = u(t) - u_0. \tag{5}$$

Differentiating the first equation of equation (5) with respect to time, we get

$$\Delta \dot{x}(t) = \dot{x}(t).$$

Note that x_0 is a constant, as we have assumed that x_0 does not vary with time.

Since the variation of the variable 'x' about the equilibrium value 'x_0' and the variation of the variable 'u' about the equilibrium value 'u_0' is very small, the terms '$\Delta x(t)$' and '$\Delta u(t)$' are very small and hence the higher order terms in equation (4) can be neglected. Therefore equation (4) reduces to

$$\Delta \dot{x}(t) = f(x_0, u_0) + \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_0 \\ u=u_0}} \Delta x(t) + \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_0 \\ u=u_0}} \Delta u(t). \tag{6}$$

To linearize equation (2), expand the equation using Taylor series about the equilibrium point, (u_0, x_0, y_0)

$$y(t) = h(x_0, u_0) + \left.\frac{\partial h}{\partial x}\right|_{\substack{x=x_0 \\ u=u_0}} \Delta x(t) + \left.\frac{\partial h}{\partial u}\right|_{\substack{x=x_0 \\ u=u_0}} \Delta u(t). \tag{7}$$

At the equilibrium point, both $f(x_0, u_0)$ and $h(x_0, u_0)$ are equal to zero. Therefore equations (6) and (7) reduce to

$$\Delta \dot{x}(t) = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_0 \\ u=u_0}} \Delta x(t) + \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_0 \\ u=u_0}} \Delta u(t). \tag{8}$$

$$y(t) = \left.\frac{\partial h}{\partial x}\right|_{\substack{x=x_0 \\ u=u_0}} \Delta x(t) + \left.\frac{\partial h}{\partial u}\right|_{\substack{x=x_0 \\ u=u_0}} \Delta u(t). \tag{9}$$

Equations (8) and (9) represents the linearized equations of the original nonlinear equation given by equations (1) and (2).

What if (x_0, u_0, y_0) is not an equilibrium point?

(Hint: $f(x_0, u_0)$ and $h(x_0, u_0)$ are not equal to zero.)

The above procedure is outlined with the help of the following example.

Example 1

Consider a simple pendulum. The equation of motion for simple pendulum is given by

$$\ddot{\theta}(t) + \frac{g}{l}\sin\theta(t) = 0.$$

The reduction of the above equation into a system of first order equations can be accomplished by letting

$$\theta(t) = x_1(t),$$

$$\dot{\theta}(t) = x_2(t).$$

The first order equations of motion are given as

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= -\frac{g}{l}\sin(x_1(t)). \end{aligned} \tag{8}$$

To obtain the equilibrium point, the derivative terms are equated to zero. In other words

$$\begin{aligned} &\dot{x}(t) = 0 \\ &\Rightarrow x_2(t) = 0 \\ &\dot{x}_2(t) = 0 \\ &\Rightarrow -\frac{g}{l}\sin(x_1(t)) = 0 \\ &\Rightarrow \sin(x_1(t)) = 0 \\ &\Rightarrow x_1(t) = n\pi \end{aligned}$$

It can be seen that the equilibrium points do not vary with time and it can be also concluded that there are infinite equilibrium values for x_1. Usually the first equilibrium value is chosen, i.e., the equilibrium value at n = 0 is chosen. Therefore the equilibrium point is given by

$$x_{10} = 0$$

$$x_{20} = 0.$$

Define

$$\Delta x_1(t) = x_1(t) - x_{10} = x_1(t),$$
$$\Delta x_2(t) = x_2(t) - x_{20} = x_2(t).$$

From equations (8), it can be seen that, the first equation is linear in nature. So this equation need not be linearized. The second equation however contains a nonlinear term, 'sin(x_1(t)'. So expanding the second equation of equation (8) using the Taylor series about the equilibrium point $x_1 = 0$ and $x_2 = 0$, we get

$$\Delta \dot{x}_2(t) = -\frac{g}{l}\sin(x_1(t))$$
$$= -\frac{g}{l}\sin(x_{10}) + \frac{d}{dx_1}(-\frac{g}{l}\sin(x_1(t)))\Big|_{x_1 = x_{10}} (x_1(t) - x_{10})$$
$$= 0 - \frac{g}{l}\cos(x_1(t))\Big|_{x_1 = x_{10}} \Delta x_1(t)$$
$$= -\frac{g}{l}\Delta x_1(t)$$

Therefore the linearized equation of motion for the simple pendulum is given by the following two first order differential equation

$$\Delta \dot{x}_1(t) = x_2(t)$$
$$\Delta \dot{x}_2(t) = -\frac{g}{l}\Delta x_1(t).$$

General Procedure for linearization

In general the state-space model of a physical system with one input and one output is given as

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), u(t)),$$
$$y(t) = h(\mathbf{x}(t), u(t)), \quad (10)$$

where the variable '**x**' is the state vector given by

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ x_n(t) \end{bmatrix}.$$

The operating point of the system is calculated by following the procedure outlined in the previous example. Let the equilibrium point be given as $\left(x_{10}, x_{20}, ..., x_{n0}, u_0, y_0\right)$.

Let the equilibrium values of the states be represented by a vector $\mathbf{X}_0$, which is given by

$$\mathbf{X}_0 = \begin{bmatrix} x_{10} \\ x_{20} \\ \cdot \\ x_{n0} \end{bmatrix}$$

Generalizing the equation (6), i.e., if the function is dependent on 'n' variables, then the Taylor series expansion of the ith entry f_i of the function f is given as

$$f_i(\mathbf{x},u) = f_i(\mathbf{x}_0,u_0) + \left.\frac{\partial f_i}{\partial x_1}\right|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ u=u_0}} \cdot (x_1(t) - x_{10}) + \left.\frac{\partial f_i}{\partial x_2}\right|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ u=u_0}} \cdot (x_2(t) - x_{20}) + \cdots + \left.\frac{\partial f_i}{\partial x_n}\right|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ u=u_0}} \cdot (x_n(t) - x_{n0})$$

$$+ \left.\frac{\partial f_i}{\partial u}\right|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ u=u_0}} \cdot (u(t) - u_0)$$

$$= \begin{bmatrix} \left.\frac{\partial f_i}{\partial x_1}\right|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ u=u_0}} & \left.\frac{\partial f_i}{\partial x_2}\right|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ u=u_0}} & \cdot & \left.\frac{\partial f_i}{\partial x_n}\right|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ u=u_0}} \end{bmatrix} \begin{bmatrix} (x_1(t) - x_{10}) \\ (x_2(t) - x_{20}) \\ \cdot \\ (x_n(t) - x_{n0}) \end{bmatrix} + \left.\frac{\partial f_i}{\partial u}\right|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ u=u_0}} \cdot (u(t) - u_0).$$

Even in the above equation, the higher order terms are neglected.

Therefore the Linearized state space equation is given by

$$\Delta \dot{\mathbf{x}}(t) = \begin{bmatrix} \left.\frac{\partial f_1}{\partial x_1}\right|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ u=u_0}} & \left.\frac{\partial f_1}{\partial x_2}\right|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ u=u_0}} & \cdot & \left.\frac{\partial f_1}{\partial x_n}\right|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ u=u_0}} \\ \left.\frac{\partial f_2}{\partial x_1}\right|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ u=u_0}} & \left.\frac{\partial f_2}{\partial x_2}\right|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ u=u_0}} & \cdot & \left.\frac{\partial f_2}{\partial x_n}\right|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ u=u_0}} \\ \cdot & \cdot & \cdot & \cdot \\ \left.\frac{\partial f_n}{\partial x_1}\right|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ u=u_0}} & \left.\frac{\partial f_n}{\partial x_2}\right|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ u=u_0}} & \cdot & \left.\frac{\partial f_n}{\partial x_n}\right|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ u=u_0}} \end{bmatrix} \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \\ \cdot \\ \Delta x_n(t) \end{bmatrix} + \begin{bmatrix} \left.\frac{\partial f_1}{\partial u}\right|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ u=u_0}} \\ \left.\frac{\partial f_2}{\partial u}\right|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ u=u_0}} \\ \cdot \\ \left.\frac{\partial f_n}{\partial u}\right|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ u=u_0}} \end{bmatrix} \Delta u(t) \qquad (11)$$

The same procedure can be followed to linearize the second equation of equation (10), which is given as

$$y(t) = \begin{bmatrix} \left.\frac{\partial h_1}{\partial x_1}\right|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ u=u_0}} & \left.\frac{\partial h_1}{\partial x_2}\right|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ u=u_0}} & . & \left.\frac{\partial h_1}{\partial x_n}\right|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ u=u_0}} \\ \left.\frac{\partial h_2}{\partial x_1}\right|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ u=u_0}} & \left.\frac{\partial h_2}{\partial x_2}\right|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ u=u_0}} & . & \left.\frac{\partial h_2}{\partial x_n}\right|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ u=u_0}} \\ . & . & . & . \\ \left.\frac{\partial h_n}{\partial x_1}\right|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ u=u_0}} & \left.\frac{\partial h_n}{\partial x_2}\right|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ u=u_0}} & . & \left.\frac{\partial h_n}{\partial x_n}\right|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ u=u_0}} \end{bmatrix} \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \\ . \\ \Delta x_n(t) \end{bmatrix} + \begin{bmatrix} \left.\frac{\partial h_1}{\partial u}\right|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ u=u_0}} \\ \left.\frac{\partial h_2}{\partial u}\right|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ u=u_0}} \\ . \\ \left.\frac{\partial h_n}{\partial u}\right|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ u=u_0}} \end{bmatrix} \Delta u(t). \qquad (12)$$

Equation (11) and equation (12) together represent the linearized state-space model of the physical system. They can be represented as

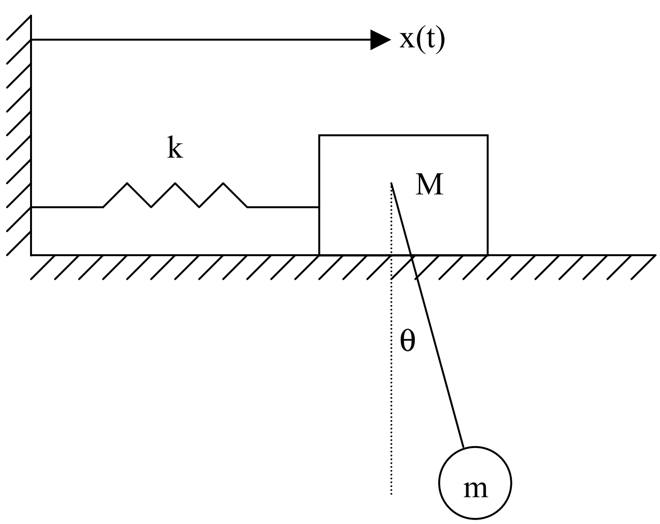
$$\Delta \dot{\mathbf{x}}(t) = \mathbf{A}\Delta\mathbf{x}(t) + \mathbf{B}\Delta u(t)$$
$$y(t) = \mathbf{C}\Delta\mathbf{x}(t) + \mathbf{D}\Delta u(t)$$

The coefficient matrices multiplying the states are known as the **Jacobian** matrices of the system.

The general linearization procedure is further explained with the help of the following example.

Example 2

Consider the following system.



The governing differential equation of motion for the above system, when written in matrix format is given as

$$\begin{bmatrix} m & ml(\cos\theta\ (t)+\sin\theta\ (t)\tan\theta\ (t)) \\ (M+m) & ml\cos\theta\ (t) \end{bmatrix}\begin{bmatrix} \ddot{x}(t) \\ \ddot{\theta}\ (t) \end{bmatrix}=\begin{bmatrix} -mg\tan\theta\ (t) \\ ml\sin\theta\ (t)\dot{\theta}^{2}\ (t)-kx(t) \end{bmatrix} \tag{13}$$

The above equation is a second order differential equation. Let

$x(t)=x_1(t),$

$\dot{x}(t)=x_2(t),$

$\theta\ (t)=x_3(t),$

$\dot{\theta}\ (t)=x_4(t).$

Using the above relations, reducing equation (13) to a first order one, we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & m & 0 & ml[\cos(x_3(t))+\sin(x_3(t))\tan(x_3(t))] \\ 0 & 0 & 1 & 0 \\ 0 & (M+m) & 0 & ml\cos(x_3(t)) \end{bmatrix}\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix}=\begin{bmatrix} x_2(t) \\ -mg\tan(x_3(t)) \\ x_4(t) \\ ml\sin(x_3(t))(\dot{x}_4{}^{2}(t))-kx_1(t) \end{bmatrix}$$

The above equation is of the form

$$\mathbf{A}\,\dot{\mathbf{x}}(t)=\mathbf{B} \tag{14}$$

It can be seen that equation (13) is non-linear because of the trigonometric terms. To linearize the equation, first the equilibrium point is obtained. Equating the derivative terms to zero, we get

$$\begin{aligned} &\dot{\mathbf{x}}(t)=0 \\ &\Rightarrow x_{10}=x_{20}=x_{30}=x_{40}=0. \end{aligned} \tag{15}$$

Let the equilibrium point be represented by a vector $\mathbf{X}_0$, given by

$$\mathbf{X}_0=\begin{bmatrix} x_{10} \\ x_{20} \\ x_{30} \\ x_{40} \end{bmatrix}.$$

Note that, the fourth element in the right hand side vector has a $\dot{x}_4$ term. Even this term has to be equated to zero. In other words to find the operating point, all the derivative terms have to be equated to zero. Equation (14) can be further reduced to the form

$$\dot{\mathbf{x}}(t) = \mathbf{A}^{-1}\mathbf{B}. \tag{16}$$

To linearize equation (16), apply Taylor's theorem about the equilibrium point and we get,

$$\Delta\dot{\mathbf{x}}(t) = (\mathbf{A}^{-1}\mathbf{B})\Big|_{\mathbf{X}=\mathbf{X}_0} + \frac{\partial}{\partial \mathbf{X}}(\mathbf{A}^{-1}\mathbf{B})\Big|_{\mathbf{X}=\mathbf{X}_0} \Delta\mathbf{x}(t) \tag{17}$$

where $\mathbf{X}_0$ is the equilibrium point of the system. Since at the equilibrium point $\mathbf{X}_0$, $\dot{\mathbf{x}}(t)$ is equal to zero, from equation (16) $(\mathbf{A}^{-1}\mathbf{B})\big|_{\mathbf{X}=\mathbf{X}_0} = 0$. Therefore equation (17) reduces to

$$\begin{aligned}\Delta\dot{\mathbf{x}}(t) &= \frac{\partial}{\partial \mathbf{X}}(\mathbf{A}^{-1}\mathbf{B})\Big|_{\mathbf{X}=\mathbf{X}_0} \Delta\mathbf{x}(t) \\ &\Rightarrow \left[\mathbf{A}^{-1}\Big|_{\mathbf{X}=\mathbf{X}_0} \frac{\partial}{\partial \mathbf{X}}(\mathbf{B})\Big|_{\mathbf{X}=\mathbf{X}_0} + \frac{\partial}{\partial \mathbf{X}}(\mathbf{A}^{-1})\Big|_{\mathbf{X}=\mathbf{X}_0} \mathbf{B}\Big|_{\mathbf{X}=\mathbf{X}_0}\right]\Delta\mathbf{x}(t).\end{aligned} \tag{18}$$

From equation (14), it can be concluded that, at the equilibrium point $\mathbf{X}_0$, since $\dot{\mathbf{x}} = 0$, $\mathbf{B}$ is equal to zero at the equilibrium point. Therefore equation (18) reduces to

$$\Delta\dot{\mathbf{x}}(t) = \mathbf{A}^{-1}\Big|_{\mathbf{X}=\mathbf{X}_0} \frac{\partial}{\partial \mathbf{X}}(\mathbf{B})\Big|_{\mathbf{X}=\mathbf{X}_0} \Delta\mathbf{x}(t).$$

The value of the matrix '**A**', at the equilibrium point is given by

$$\mathbf{A}(\mathbf{X}_0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & m & 0 & ml \\ 0 & 0 & 1 & 0 \\ 0 & (M+m) & 0 & ml \end{bmatrix}$$

$$\frac{\partial \mathbf{B}}{\partial \mathbf{X}} = \left[\frac{\partial \mathbf{B}}{\partial x_1}\frac{\partial \mathbf{B}}{\partial x_2}\frac{\partial \mathbf{B}}{\partial x_3}\cdots\cdots\frac{\partial \mathbf{B}}{\partial x_n}\right]$$

Evaluating each of the above at the equilibrium position we get,

$$\frac{\partial \mathbf{B}}{\partial x_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -k \end{bmatrix}, \frac{\partial \mathbf{B}}{\partial x_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \frac{\partial \mathbf{B}}{\partial x_3} = \begin{bmatrix} 0 \\ -mg\sec^2(x_3(t)) \\ 0 \\ -ml\cos(x_3(t))(\dot{x}_4{}^2(t)) \end{bmatrix}$$

$$\frac{\partial \mathbf{B}}{\partial x_4} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2ml\sin(x_3(t))\dot{x}_4(t) \end{bmatrix}$$

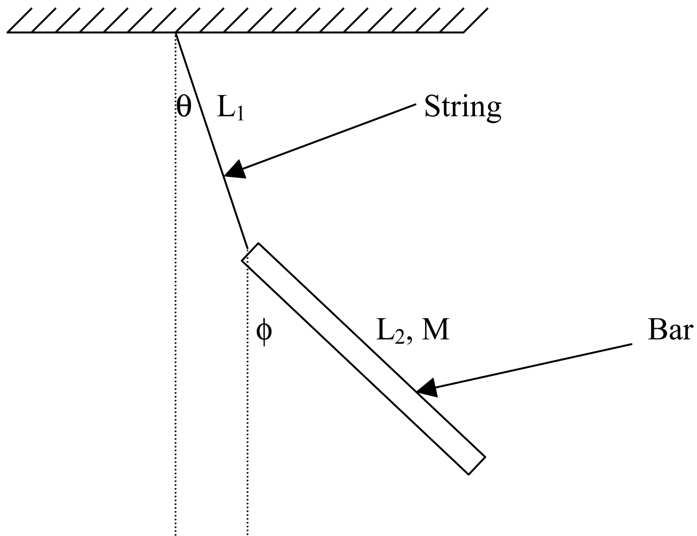
$$\left.\frac{\partial \mathbf{B}}{\partial \mathbf{X}}\right|_{\mathbf{X}=\mathbf{X}_0} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -mg & 0 \\ 0 & 0 & 0 & 1 \\ -k & 0 & 0 & 0 \end{bmatrix}$$

Therefore the linearized equation is given by

$$\Delta \dot{\mathbf{x}}(t) = \mathbf{A}^{-1}(\mathbf{X}_0)\left.\frac{\partial \mathbf{B}}{\partial \mathbf{X}}\right|_{\mathbf{X}=\mathbf{X}_0} \Delta \mathbf{x}(t)\,.$$

Assignment

For the system shown below



a) Derive the governing differential equations of motion.

b) Find the operating point or the equilibrium point of the system.

c) Linearize the differential equation about the equilibrium point.

(Note: Assume that the string is always taut.)

Recommended reading

"Feedback Control of Dynamic Systems" 4th Edition, by Gene F. Franklin et.al – pp –68-74.