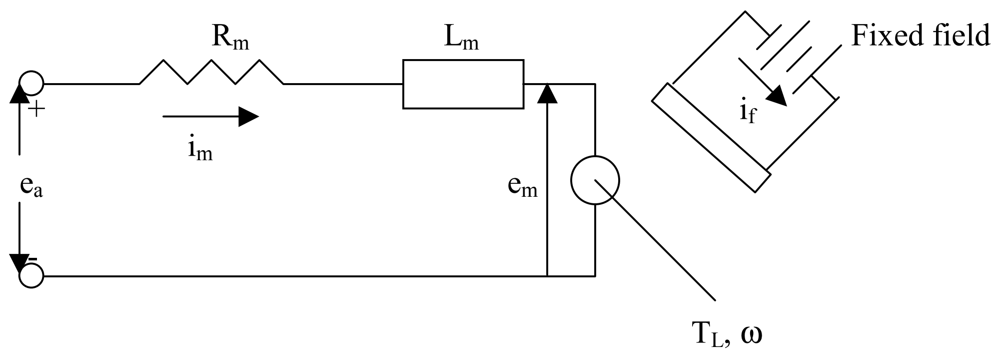
HANDOUT A.4 - MODELING EXAMPLES OF DYNAMIC SYSTEMS

Introduction

This handout consists of various modeling examples of dynamic systems. It does not cover the entire subject. It is always advisable for you to practice more such problems to become familiar with modeling of dynamic systems.

Example 1: DC Servomotor

The dc motor is one example of an electro-mechanical system. The circuit diagram of the armature controlled dc motor is given as follows:



Description of the variables used.

i_m = Armature current
e_a = Applied voltage
R_m = Armature resistance
L_m = Armature inductance
ω = Speed of the motor
e_m = Back emf
T_m = Torque developed by the motor.

Note that, the time dependence of all the variables is ignored. Unless specified explicitly, all variables are time dependent.

The developed torque of a dc motor is proportional to the magnitude of the flux due to the field current i_f and the armature current i_m. Therefore the developed torque can be expressed as

$$T_m = K_3 \phi \; i_m . \tag{1}$$

For any given motor, the only two adjustable quantities are the flux and the armature current. There are two modes of operation of a servomotor. For the armature-controlled mode (circuit diagram shown above), the filed current is held constant and an adjustable voltage is applied to the armature. In the field control mode, the armature current is held constant and a voltage is applied to the field circuit. Since the above circuit is armature controlled, the filed current is held constant and therefore the equation (1) can be represented as

$$T_m = K_T i_m , \tag{2}$$

where K_T is called the motor torque constant.

When the motor armature is rotating, a voltage e_m is induced that is proportional to the product of the flux and the speed of the motor. Since the polarity of this voltage opposes that of the applied voltage, this voltage is called the *back emf*. Since the flux is held constant, the induced voltage is proportional to the speed of the motor ω_m. This can be represented as

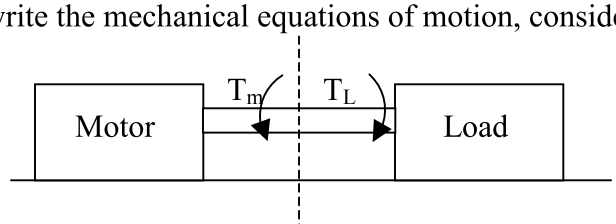
$$e_m = K_1\phi\ \omega = K_b\omega_m = K_b \frac{d\theta}{dt} , \tag{3}$$

where K_b is called the generator constant.

From the circuit diagram shown above, the circuit loop equation can be written as

$$e_a = L_m \frac{di_m}{dt} + R_m i_m + e_m \tag{4}$$

To write the mechanical equations of motion, consider the diagram shown below.



Let 'J' be the inertia of the entire system and 'B' be the damping constant. Then writing the torque balance equation, we get

$$T_m - T_L = J\frac{d\omega}{dt} + B\omega = J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} \tag{5}$$

Combining equations (2) and (5), we have

$$J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = K_T i_m - T_L \tag{6}$$

Similarly combining equation (4) and equation (3), we get

$$e_a = L_m \frac{di_m}{dt} + R_m i_m + K_b \frac{d\theta}{dt}. \tag{7}$$

Equations (6) and (7) represents the governing equations of motion of the dc motor. Let the individual states of the system be given as

$$\begin{aligned} x_1 &= \theta \\ x_2 &= \omega = \frac{d\theta}{dt} \\ x_3 &= i_m \end{aligned} \tag{8}$$

From the first two equations of equation (8), we get

$$\dot{x}_1 = x_2. \tag{9}$$

From equation (8) and equation (6), we have

$$\begin{aligned} &J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = K_T i_m - T_L \\ &\Rightarrow J\dot{x}_2 + Bx_2 = K_T x_3 - T_L \\ &\Rightarrow \dot{x}_2 = -\frac{B}{J}x_2 + \frac{K_T}{J}x_3 - T_L \end{aligned} \tag{10}$$

From Equation (8) and equation (7), we get

$$\begin{aligned} &e_a = L_m \frac{di_m}{dt} + R_m i_m + K_b \frac{d\theta}{dt} \\ &\Rightarrow e_a = L_m \dot{x}_3 + R_m x_3 + K_b x_2 \\ &\Rightarrow \dot{x}_3 = -\frac{K_b}{L_m}x_2 - \frac{R_m}{L_m}x_3 + \frac{1}{L_m}e_a \end{aligned} \tag{11}$$

Combining equations (9), (10) and (11) and representing them in matrix format, we get

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{B}{J} & \frac{K_T}{J} \\ 0 & -\frac{K_b}{L_m} & -\frac{R_m}{L_m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L_m} \end{bmatrix} e_a + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} T_L \tag{12}$$

The above equation is in the form of

$$\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{B}u + \mathbf{G}w.$$

Where 'w' is the disturbance to the system.
From equation (12), it can be concluded that the applied voltage 'e_a' is the input to the system. The developed torque by the dc motor is the output to the system. Representing equation (2) in matrix format, we have

$$T_m = K_T i_m = \begin{bmatrix} 0 & 0 & K_T \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{13}$$

Equations (12) and (13) represent the state space model of an armature controlled dc motor.

Characteristics of a DC motor
Combining equations (3) and (4), we get

$$\begin{aligned} e_m &= e_a - R_m i_m - L_m \frac{di_m}{dt} \\ \Rightarrow K_b \omega_m &= e_a - R_m i_m - L_m \frac{di_m}{dt} \end{aligned} \tag{14}$$

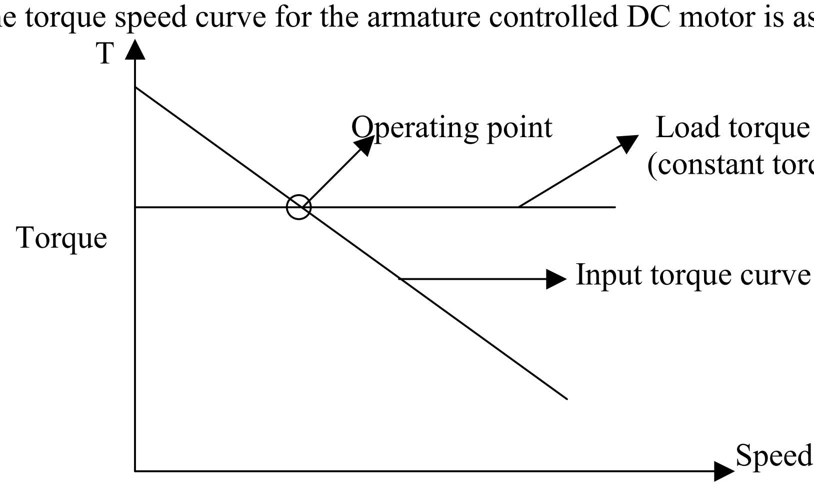
But we know that from equation (2) that $T_m = K_T i_m$. Substituting the value of 'i_m' in equation (14), we have

$$K_b \omega_m = e_a - R_m \frac{T_m}{K_T} - L_m \frac{di_m}{dt}. \tag{15}$$

Assuming a very low inductance, the equation (15) can be rewritten as

$$\omega_m = K_1 e_a - K_2 T_m \tag{16}$$

The torque speed curve for the armature controlled DC motor is as shown below.



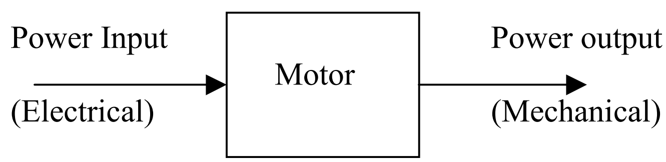
Torque-speed curve of a DC motor

Load matching

The things that are usually regarded as important about an electric motor are its maximum speed and maximum power output. Another is the motor's power and torque characteristic, which are often overlooked, but these need to be considered carefully because the torque and the power characteristics determine whether or not the motor can drive the attached load correctly. To illustrate, a motor driving a fan might require the same power output as a motor driving a conveyor belt. However, the torque and power characteristics required of the motor would be completely different. To be able to successfully match a motor to a load we need to consider carefully the characteristics of the load. In other words, in the torque-speed curve of the motor, the load torque and the input torque curves must intersect. The point of intersection represents the condition at which the motor tends to operate.

There are different types of load, each giving different characteristics and to select the correct motor, the knowledge of the load profile is essential. For example the most commonly found in the industry is the quadratic torque load. In this case the torque varies as the square of the speed, whereas the power varies as the cube of the speed. This is the typical torque and speed characteristics of a fan or a pump.

Consider the following diagram



From the above figure it can be seen that, the power input must be equal to the power output. But since the Motor does not operate at its full efficiency, the following relation is obtained.

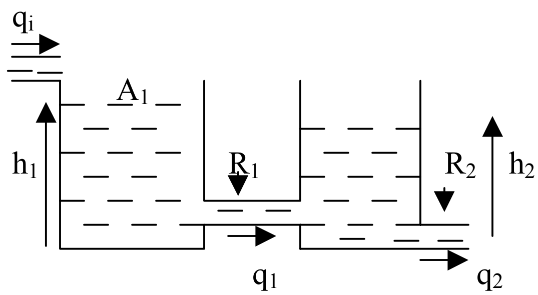
$$P_{in}^{Elec} = \frac{P_{out}^{mech}}{\eta}.$$

But since the power output is equal to the product of the load torque and the speed of the motor,

$$P_{in}^{Elec} = \frac{T_L.\omega_m}{\eta}.$$

From the above relation it can be concluded that, for a rated speed and voltage of the motor, there is a fixed amount of load torque that the motor can drive.

Example 2: A Liquid-Level system



The above figure represents a two-tank liquid level system.

Definitions of the system parameters

q_i, q_1, q_2 = Flow rates of fluid
h_1, h_2 = Heights of the fluid level is the tanks
R_1, R_2 = Flow resistance
A_1, A_2 = Cross-sectional area of the tanks.

The basic linear relationship between the flow rate 'q', change in the height, 'h' and the resistance to the flow, 'r' is given by

$q = \frac{h}{r}$ = flow rate through orifice.

The mass balance equation of the system can be written as,

Rate of fluid storage in the tank = (tank input flow rate) – (tank output flow rate) = net flow rate = $A\frac{dh}{dt}$

Applying the above two relations to tanks 1 and 2, we have

$$\begin{aligned} A_1\frac{dh_1}{dt} &= q_i - q_1 = q_i - \frac{h_1 - h_2}{R_1} \\ A_2\frac{dh_2}{dt} &= q_1 - q_2 = \frac{h_1 - h_2}{R_1} - \frac{h_2}{R_2} \end{aligned} \quad (14)$$

Let the individual states of the system be

$$\begin{aligned} x_1 &= h_1 \\ x_2 &= h_2 \end{aligned} \quad (15)$$

The levels of the two tanks are the output of the system, i.e., $y_1 = h_1$ and $y_2 = h_2$.

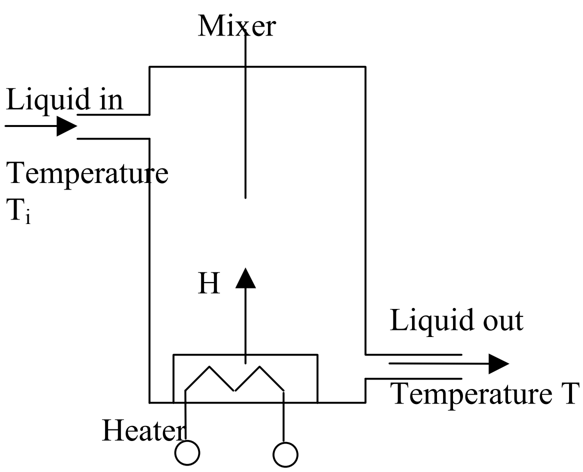
From equations (14) and (15), the state space model of the system can be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_1 A_1} & \frac{1}{R_1 A_1} \\ \frac{1}{R_1 A_2} & -\frac{1}{R_1 A_2} - \frac{1}{R_2 A_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{A_1} \\ 0 \end{bmatrix} q_i$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Example 3: A Thermal System

Consider the simple thermal system shown below.



Assume that the tank is thermally insulated and the liquid in the tank is kept at uniform temperature by perfect mixing with the help of a mixer. Assume that the steady state temperature of the incoming fluid is T_i and that of the out flowing liquid is T. The steady state thermal input rate from the heater is H and the liquid flow rate is assumed to be constant.

Let ΔH be a small increase in the thermal input rate from its steady state value. This increase in thermal input will result in the increase in the thermal output flow rate by an amount ΔH_1 and an increase in the thermal storage rate of the liquid in the tank by an amount ΔH_2. Consequently, the temperature of the liquid in the tank and the out flowing liquid rises by ΔT. Since the insulation is perfect, the increase in the thermal output flow rate is only due to the rise in temperature of the out flowing liquid and is given by

$$\Delta H_1 = m C_p \Delta T \quad (16)$$

where 'm' is the liquid flow rate and 'C_p' is the specific heat of the liquid.

Let us define the thermal resistance as

$$R = \frac{1}{mC_p}.$$

Therefore equation (16) reduces to

$$\Delta H_1 = \frac{\Delta T}{R} \tag{17}$$

The rate of heat storage in the tank is given by

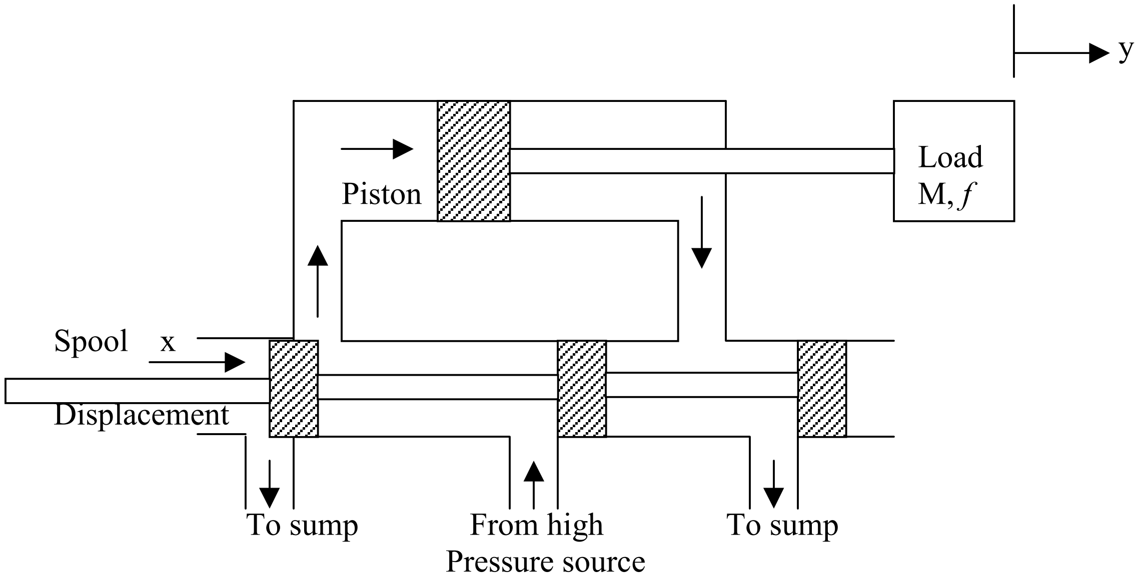
$$\Delta H_2 = MC_p \frac{d(\Delta T)}{dt} = C \frac{d(\Delta T)}{dt} \tag{18}$$

where 'M' is the mass of the liquid in the tank, $\frac{d(\Delta T)}{dt}$ is the rate of rise of temperature in the tank and 'C' which is equal to the product of 'M' and 'C_p' is called the thermal capacitance. Therefore the heat flow balance equation or the energy balance equation can be stated as, ' the thermal increase in the input must be equal to the sum of the thermal increase in the output and the thermal increase of the liquid stored in the tank'. For the above system the energy balance equation is given by

$$\begin{aligned} \Delta H &= \Delta H_1 + \Delta H_2 \\ \Delta H &= \frac{\Delta T}{R} + C \frac{d(\Delta T)}{dt} \\ \Rightarrow R(\Delta H) &= \Delta T + RC \frac{d(\Delta T)}{dt} \end{aligned} \tag{19}$$

The third equation of equation (19) describes the dynamics of the thermal system with the assumption that the temperature of the incoming fluid is constant.

Example 4: A Hydraulic system



The above figure shows a simple hydraulic actuator, in which the motion of spool regulates the flow of oil to either side of the power cylinder. When the spool moves to the right, the oil from the high-pressure source enters into the power cylinder, on the left of the power piston. This creates a differential pressure across the piston, which causes the power piston to move to the right, pushing the oil in front of it to the sump. The oil is pressurized by a pump and is recirculated in the system. The load rigidly coupled to the piston moves a distance 'y' from its reference position in response to the displacement 'x' of the valve spool from its neutral position.

There exists a nonlinear relationship between the volumetric oil flow rate 'q' into the power piston and the differential pressure 'Δp' across the piston for small values of spool displacement 'x'. The relationship between 'q', 'x' and 'Δp' may be written as

$$q = f(x, \Delta p). \tag{20}$$

Expanding the above equation into Taylor's series about the normal operating point $(q_0, \Delta p_0, x_0)$ and neglecting all the terms of second and higher derivatives, we get

$$q = q_0 + \left.\frac{\partial q}{\partial x}\right|_{\substack{x=x_0 \\ \Delta p=\Delta p_0}} (x - x_0) + \left.\frac{\partial q}{\partial \Delta p}\right|_{\substack{x=x_0 \\ \Delta p=\Delta p_0}} (\Delta p - \Delta p_0) \tag{21}$$

For this system, the normal operating point corresponds to $q_0 = 0$, $\Delta p_0 = 0$, $x_0 = 0$, therefore the equation (21) reduces to

$$q = K_1 x - K_2 \Delta p$$

where

$$\mathrm{K}_1 = \left.\frac{\partial q}{\partial x}\right|_{\substack{x=0\\ \Delta p=0}}; \qquad K_2 = -\left.\frac{\partial q}{\partial \Delta p}\right|_{\substack{x=0\\ \Delta p=0}}. \tag{22}$$

Equation (22) gives a linearized relationship among ‘q’, ‘x’ and ‘Δp’.

Assuming leakage and compressibility flows to be negligible, the rate of oil flow into the piston is proportional to the rate, at which the piston moves, i.e.,

$$q = A\frac{dy}{dt} = A\dot{y} \tag{23}$$

where A is the area of the piston.

The force on the piston is ‘Ap’ which moves the load consisting of mass ‘M’ and viscous friction with coefficient ‘*f*’. Writing the Newton’s second law motion (or the force balance equation), we have

$$A\Delta p = M\ddot{y} + f\dot{y} \tag{24}$$

From equations (22) and (23), we get

$$\frac{A}{K_2}\left(K_1 x - q\right) = M\ddot{y} + f\dot{y}. \tag{25}$$

Substituting the value of ‘q’ from equation (23) in equation (25), we have

$$\frac{A}{K_2}(K_1 x - A\dot{y}) = M\ddot{y} + f\dot{y}$$
$$\Rightarrow \frac{AK_1}{K_2}x = M\ddot{y} + (f + \frac{A^2}{K_2})\dot{y} \tag{26}$$

Taking the Laplace transform of the second equation of equation (26), we get

$$\frac{AK_1}{K_2}X(s) = Ms^2Y(s) + (f + \frac{A^2}{K_2})sY(s)$$
$$\Rightarrow \frac{Y(s)}{X(s)} = \frac{AK_1/K_2}{\left(Ms^2 + (f + \frac{A^2}{K_2})s\right)} \tag{27}$$

The second equation of equation (27) represents the transfer function between the input 'x', which is the displacement of the spool and the output 'y', which is the displacement of the load attached to the piston.

The second equation of equations (26) represents the governing differential equation of the system. To denote the equations in the state-space form, let the states be defined as

$$\begin{aligned} y &= x_1 \\ \dot{y} &= x_2 \end{aligned} \tag{28}$$

From the above relations it can be concluded that

$$\dot{x}_1 = x_2. \tag{29}$$

Substituting the relations given by equation (28) in the second equation of equation (27), we get

$$\begin{aligned} &\frac{AK_1}{K_2}x = M\ddot{y} + (f + \frac{A^2}{K_2})\dot{y} \\ &\Rightarrow \frac{AK_1}{K_2}x = M\dot{x}_2 + (f + \frac{A^2}{K_2})x_2 \\ &\Rightarrow \dot{x}_2 = -\frac{1}{M}(f + \frac{A^2}{K_2})x_2 + \frac{AK_1}{MK_2}x \end{aligned} \tag{30}$$

Representing the equation (29) and the third equation of equations (30) in the matrix format, we have

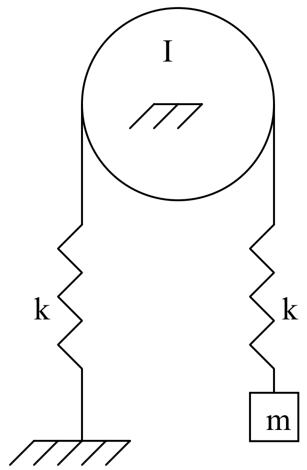
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{M}(f + \frac{A^2}{K_2}) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{AK_1}{MK_2} \end{bmatrix} x \tag{31}$$

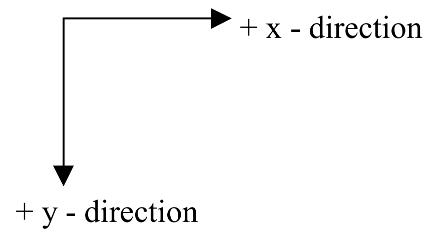
The output of the system is the displacement of the load, 'y', for the given input 'x'. Therefore representing the output equation in matrix format, we get

$$\begin{aligned} &Y = y \\ &\Rightarrow Y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned} \tag{32}$$

Equation (31) and the second equation of equation (32) represent the state-space model of the hydraulic model discussed above.

Example 5: A Mechanical system





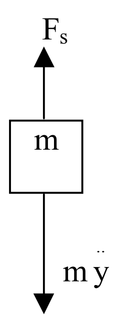
The Disc with mass moment of inertia ‘I’ rotates in the counterclockwise direction. The block of mass ‘m’ moves a distance of ‘y’ units from the static equilibrium position in the positive y-direction.

From the above figure, it can be seen that the system has two degrees of freedom. One is the rotation of the disc and the other the linear displacement of the block. Let the two degrees of freedom be represented as ‘θ’ and ‘y’, respectively.

The next step is to determine the velocity and acceleration components of the block and disc. The linear velocity and the linear acceleration of the block are given by ‘ $\dot{y}$ ’ and ‘ $\ddot{y}$ ’ respectively. Similarly the angular velocity and angular acceleration of the disc are given by ‘ $\dot{\theta}$ ‘ and ‘ $\ddot{\theta}$ ‘ respectively. This stage is called the kinematics stage.

The next step is to draw the free body diagram of the block and the disc. This stage is called the kinetics stage.

Free body diagram of the block



Note that the gravity force is not considered in the free body diagram. The reason for this is that 'y' is considered from the static equilibrium position and hence the spring force at the equilibrium position is cancelled by the weight of the block.

Writing the Newton's second law of motion for the block, which states that sum of all the forces acting on the block must be equal to the product of its mass and acceleration.

$$\sum F = ma$$

$$\Rightarrow -F_s = m\ddot{y} \qquad (33)$$

$$\Rightarrow m\ddot{y} + F_s = 0$$

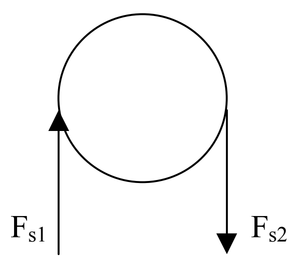
Since the disc is rotating in the counter clockwise direction, there is an elongation in the right hand spring by an amount of 'Rθ' units. The block is assumed to move down, which in turn will again produce an elongation in the right hand spring by an amount of 'y' units. Therefore the total elongation of the right hand spring due to the movement of the disc and the block is 'Rθ + y' units. Therefore the spring force is given by

$$F_s = k(R\theta + y)$$

Therefore the equation of motion of the block is given by

$$m\ddot{y} + k(R\theta + y) = 0 \qquad (34)$$

Free body diagram of the disc



Taking moments about the center of the disc, we get

$$I\ddot{\theta} = -F_{s1}R - F_{s2}R$$

$$\Rightarrow I\ddot{\theta} = -k(R\theta)R - k(R\theta + y)R \qquad (35)$$

$$\Rightarrow I\ddot{\theta} + kR^2\theta + kR(R\theta + y) = 0$$

The third equation of equation (35) and equation (34) together represent the governing differential equation of motion for the system defined.

To represent the above-derived differential equations in state-space form, the states of the system have to be defined. Let the states be given by

$$
\begin{aligned}
y &= x_1 \\
\dot{y} &= x_2 \\
\theta &= x_3 \\
\dot{\theta} &= x_4.
\end{aligned}
\tag{36}
$$

From the above relations the following two state equations can be derived

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_3 &= x_4
\end{aligned}
\tag{37}
$$

Substituting the relations given by equation (36) in equation (34), we get

$$
\begin{aligned}
& m\ddot{y} + k(R\theta + y) = 0 \\
& \Rightarrow m\dot{x}_2 + k(Rx_3 + x_1) = 0 \\
& \Rightarrow \dot{x}_2 = -\frac{k}{m}x_1 - \frac{kR}{m}x_3.
\end{aligned}
\tag{38}
$$

Similarly substituting the relations given by equation (36) in the third equation of equations (35), we have

$$
\begin{aligned}
& I\ddot{\theta} + kR^2\theta + kR(R\theta + y) = 0 \\
& \Rightarrow I\dot{x}_4 + kR^2x_3 + kR(Rx_3 + x_1) = 0 \\
& \Rightarrow \dot{x}_4 = -\frac{kR}{I}x_1 - \frac{2kR^2}{I}x_3.
\end{aligned}
\tag{39}
$$

Representing the equations (37), the third equation of equations (38) and the third equation of equations (39) in the matrix format, we have

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{m} & 0 & -\frac{kR}{m} & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{kR}{I} & 0 & -\frac{2kR^2}{I} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}
\tag{40}
$$

If the output of the system is the displacement of the block, then representing the output relation in the matrix format, we get

$$Y = y$$

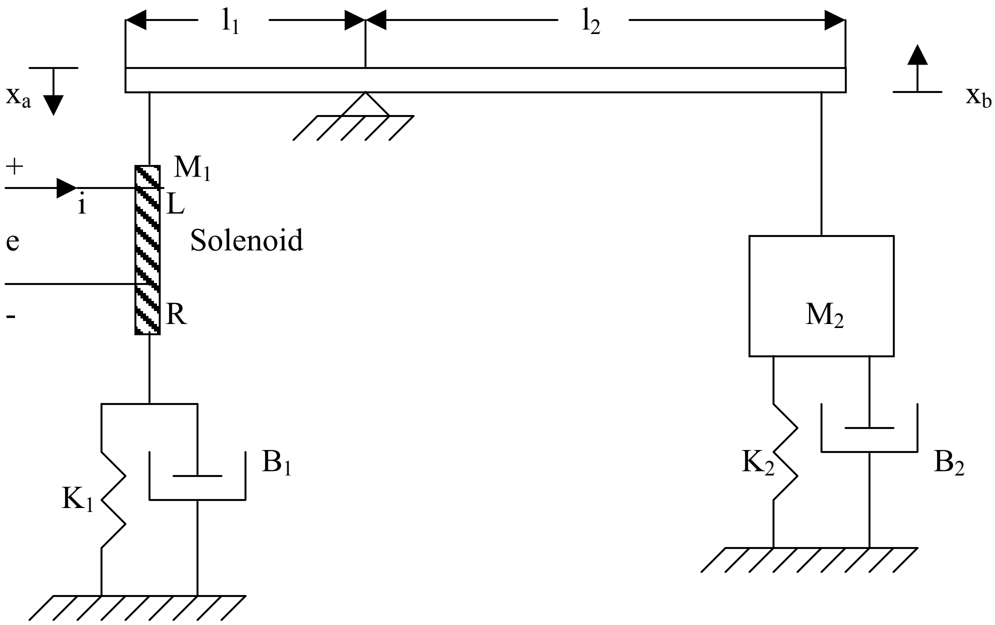
$$\Rightarrow Y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{41}$$

Equation (40) and equation (41) represent the state-space representation of the above system.

Assignment

1) An electro-mechanical actuator contains a solenoid, which produces a magnetic force proportional to the current in the coil, $f = K_i i$. The coil ahs resistance and inductance.

a) Write the differential equations of performance.
b) Write the state equations.



2) Problems 2.1, 2.3, 2.8, 2.20, 2.22 in “Feedback Control of Dynamic Systems” 4th Edition, by Gene F. Franklin et.al.

Recommended reading

“Feedback Control of Dynamic Systems” 4th Edition, by Gene F. Franklin et.al – pp 22-68.