**Problem 1**

The asymptotes of the Bode plot magnitude of a **stable minimum-phase** system are given in the following sketch. Slopes are given in dB per decade. Write out the transfer function of this system.

![Bode Plot Sketch](image)

**Note:** Do not try to compute the value of the damping coefficient.

**SOLUTION**

The first part of the Bode plot comes down from infinity at -40 dB/dec which is the plot of \( \frac{1}{s^2} \). Note that the line crosses the 0dB line (i.e., gain of 1) at 1 rad/s.

At 2 rad/s, a slope of 20 dB/dec is added. Since the breakpoint is 2 rad/s the function this describes is \( \frac{s}{2} + 1 \).
At 5 rad/s an additional 20 dB/dec is added. \( \Rightarrow \frac{s}{5} + 1 \)

At 20 rad/s 20 dB/dec is subtracted \( \Rightarrow \frac{1}{s/20 + 1} \)

At 100 rad/s 40 dB/dec is subtracted \( \Rightarrow \frac{1}{(s/100)^2 + 2\zeta s/100 + 1} \)

So the final transfer function is:

\[
G(s) = \frac{k \left( \frac{s}{2} + 1 \right) \left( \frac{s}{5} + 1 \right)}{s^2 \left( \frac{s}{20} + 1 \right) \left( \frac{s}{100} \right)^2 + 2\zeta s/100 + 1}
\]

To verify that the gain is about one at 1 rad/s:

\[
|G(1)| = \left| \frac{k \left( \frac{j}{2} + 1 \right) \left( \frac{j}{5} + 1 \right)}{j^2 \left( \frac{j}{20} + 1 \right) \left( \frac{j}{100} \right)^2 + 2\zeta j/100 + 1} \right| = k \ast 1.14 \approx 1 \Rightarrow k \sim 0.9
\]

Use software to plot the bode plot assuming a gain of 1 and damping ratio of 1 to verify solution.
Problem 2

Identify the pole locations of the system represented by the following Bode plot:

SOLUTION

Since the high frequency roll-off is -40 dB/dec, this system is a second order system. The corner frequency is about $\omega_1 = 3.3$ rad/s. At this frequency, there is no or a very small resonance peak, so $\zeta \cong 1$. From the phase plot we can see this system is unstable. Thus a candidate transfer function of this system is $\frac{1}{\left(\frac{s}{3.3} - 1\right)^2}$. There is a double pole at $s = 3.3$ rad/s.
Solution:

\[ G(s) = \frac{100}{(s+1)^2(s^2 + 20s + 100)} = \frac{100}{(s+1)^2(s+10)^2} \approx \frac{1}{(s+1)^2 \left( \frac{s}{10} + 1 \right)^2} \]

The transfer function is converted to the following from by substituting \( s = j\omega \)

\[ G(j\omega) = \frac{1}{(j\omega + 1)^2 \left( \frac{j\omega}{10} + 1 \right)^2} \]

From here we just read off the 2 breakpoints: 1 rad/s and 10 rad/s. The DC gain is 1.
The transfer function in the numerator has no phase. The first, second order transfer function in the denominator has phase of 0 degrees at very low frequencies and -180 degrees at very high frequencies. The second, second order transfer function in the denominator has a phase of 0 degrees at low frequencies and -180 degrees at high frequencies. This means the combined system has phase of 0 degrees at very low frequencies and -360 degrees at very high frequencies.

\[ \frac{1}{(s+1)^2} \Rightarrow \phi = \begin{cases} 0^\circ & \omega \rightarrow 0 \\ -180^\circ & \omega \rightarrow \infty \end{cases} \]

\[ \frac{1}{\left(\frac{s}{10} + 1\right)^2} \Rightarrow \phi = \begin{cases} 0^\circ & \omega \rightarrow 0 \\ -180^\circ & \omega \rightarrow \infty \end{cases} \]

total \quad \phi = \begin{cases} 0^\circ & \omega \rightarrow 0 \\ -360^\circ & \omega \rightarrow \infty \end{cases}

\[
\text{sys} = \text{tf}([100],[1 22 141 220 100]);
\]

\[
\text{bode(sys)}
\]
b) First rearrange as follows:

\[
\frac{100s + 400}{(s + 100)(s^2 - 60s + 3600)} = \frac{1}{900} \frac{s/100 + 1}{(s/60)^2 - (s/60) + 1}.
\]

From here we just read off the 3 breakpoints: 4 rad/s, 60 rad/s and 100 rad/s. The DC gain is 1/900. At the natural frequency 60 rad/s the magnitude is increased by $2\zeta = 1$ since $\zeta = 1/2$.

The transfer function in the numerator has phase of 0 degrees at very low frequencies and 90 degrees at very high frequencies. The first order transfer function in the denominator has phase of 0 degrees at very low frequencies and -90 degrees at very high frequencies. The second order transfer function in the denominator has a phase of -360 degrees at low frequencies and -180 degrees at high frequencies. This means the combined system has phase of -360 degrees at very low frequencies and -180 degrees at very high frequencies.
\[ \frac{s}{4} + 1 \Rightarrow \phi = \begin{cases} 0^\circ & \omega \to 0 \\ 90^\circ & \omega \to \infty \end{cases} \]

\[ \frac{1}{s/100 + 1} \Rightarrow \phi = \begin{cases} 0^\circ & \omega \to 0 \\ -90^\circ & \omega \to \infty \end{cases} \]

\[ \frac{1}{\left(\frac{s}{60}\right)^2 - \left(\frac{s}{60}\right) + 1} \Rightarrow \phi = \begin{cases} -360^\circ & \omega \to 0 \\ -180^\circ & \omega \to \infty \end{cases} \]

**total** \[ \phi = \begin{cases} -360^\circ & \omega \to 0 \\ -180^\circ & \omega \to \infty \end{cases} \]

```matlab
sys=tf([100, 400],[1 40 -2400 360000]);
bode(sys)
```
c) First factorize and put the transfer function into standard Bode forms:

\[
\frac{4(0.5s - 1)}{s(s^2 + 20s + 64)} = \frac{4(0.5s - 1)}{s(s + 4)(s + 16)} = \frac{4}{(4)(16)} \frac{(s - 1)}{s(s + 1)(\frac{s}{4} + 1)(\frac{s}{16} + 1)}.
\]

From here we just read off the 3 breakpoints: 2 rad/s, 4 rad/s and 16 rad/s. In order to fix the plot, you can calculate the magnitude at a convenient frequency such as 1 rad/s, this way

\[
|G(j\omega)| = \left| \frac{1}{16} \frac{j^2 - 1}{j^2 + 1} \right| = \frac{1}{16} \frac{\sqrt{\left(\frac{1}{2}\right)^2 + 1}}{\left|j\right| + 1} \approx \frac{\sqrt{1.25}}{16}.
\]

In the above approximation, \(\left|\frac{j}{4} + 1\right| \approx 1\) and \(\left|\frac{j}{16} + 1\right| \approx 1\).
\[
20\log\left(\frac{\sqrt{1.25}}{16}\right)
\]

\[
\frac{1}{s} \Rightarrow \phi = \begin{cases} 
-90 & \omega \to 0 \\
-90 & \omega \to \infty 
\end{cases}
\]

\[
\frac{s}{2} - 1 \Rightarrow \phi = \begin{cases} 
180 & \omega \to 0 \\
90 & \omega \to \infty 
\end{cases}
\]

\[
\frac{1}{s+1} \Rightarrow \phi = \begin{cases} 
0 & \omega \to 0 \\
-90 & \omega \to \infty 
\end{cases}
\]

\[
\frac{s}{4} + 1 \Rightarrow \phi = \begin{cases} 
0 & \omega \to 0 \\
-90 & \omega \to \infty 
\end{cases}
\]

\[
total \Rightarrow \phi = \begin{cases} 
90 & \omega \to 0 \\
-180 & \omega \to \infty 
\end{cases}
\]

\[
sys=tf([2 -4],[1 20 64 0]);
\]

\[
bode(sys)
\]
\[
G(s) = \frac{100}{(s+1)^2(s^2+20s+100)}
\]
\[ G(s) = \frac{100s + 400}{(s + 10)(s^2 - 60s + 3600)} \]

**Graphs:**
- Frequency response
- Magnitude response
- Phase response

**Analysis:**
- DC gain: \( 20 \log \left( \frac{1}{\text{qoo}} \right) \approx -59 \) dB
- \(-59\) dB per decade
- \(-20\) dB per decade
$$G(s) = \frac{4(0.5s - 1)}{s(s^2 + 20s + 64)}$$
\[ G(f) = \frac{4(0,5 - 1)}{5 + 2005 + 64} \]