

## HANDOUT M.2 - DIFFERENTIATION AND INTEGRATION

**Section 1: Differentiation***Definition of derivative*

A derivative  $f'(x)$  of a function  $f(x)$  depicts how the function  $f(x)$  is changing at the point 'x'. It is necessary for the function to be continuous at the point 'x' for the derivative to exist. A function that has a derivative is said to be **differentiable**. In general, the derivative of the function  $y = f(x)$ , also denoted  $dy/dx$ , can be defined as

$$\frac{dy}{dx} = \frac{df(x)}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

This means that as  $\Delta x$  gets very small, the difference between the value of the function at 'x' and the value of the function at  $x + \Delta x$  divided by  $\Delta x$  is defined as the derivative.

**Derivatives of some common functions**

$$\frac{d}{dx}(x) = 1$$

$$\frac{d}{dt}(t^2) = 2t$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(\log x) = \frac{1}{x}$$

$$\frac{d}{dx}(e^{ax}) = ae^{ax}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

**General rules of differentiation**

1. The derivative of a constant is equal to zero. If  $y = c$ ,

$$\frac{dy}{dx} = \frac{d}{dx}(c) = 0$$

where 'c' is any arbitrary constant.

2. The derivative of the product of a constant and a function is equal to the constant times the derivative of the function. If  $y = cf(x)$

$$\frac{dy}{dx} = \frac{d}{dx}(cf(x)) = c \frac{df}{dx}$$

### Example 1

If  $y = 8x$ , then

$$\frac{dy}{dx} = \frac{d}{dx}(8x) = 8 \frac{d}{dx}(x) = 8(1) = 8$$

3. The derivative of the sum or difference of two functions is equal to the sum or difference of the derivatives of the functions. If  $y = f(x) \pm g(x)$

$$\frac{dy}{dx} = \frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$$

### Example 2

If  $y = 8x - x^2$ , then

$$\frac{dy}{dx} = \frac{d}{dx}(8x - x^2) = \frac{d}{dx}(8x) - \frac{d}{dx}(x^2) = 8 - 2x$$

4. **Product rule** The derivative of the product of two functions is equal to the first function times the derivative of the second, plus the second function times the derivative of the first. If  $y = f(x)g(x)$

$$\frac{dy}{dx} = \frac{d}{dx}(f(x)g(x)) = f(x) \frac{d}{dx}(g(x)) + g(x) \frac{d}{dx}(f(x))$$

### Example 3

If  $y = xe^x$ , then

$$\frac{dy}{dx} = \frac{d}{dx}(xe^x) = x \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(x) = xe^x + e^x(1) = xe^x + e^x$$

### Example 4

If  $y = x^2 \sin(x)$ , then

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^2 \sin(x)) = x^2 \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x^2) \\ &= x^2 \cos x + \sin x (2x) = x^2 \cos x + 2x \sin x\end{aligned}$$

5. **Division rule** If  $y = f(x)/g(x)$ , then

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x) \frac{d}{dx}(f(x)) - f(x) \frac{d}{dx}(g(x))}{g(x)^2}$$

### Example 5

If  $y = e^x/x$ , then

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x}{x}\right) = \frac{x \frac{d}{dx}(e^x) - e^x \frac{d}{dx}(x)}{x^2} = \frac{xe^x - e^x(1)}{x^2} = \frac{xe^x - e^x}{x^2}$$

6. **Chain rule** If  $y = f(u)$  and  $u = g(x)$ , that is, if  $y$  is a function of a function, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

7. If  $y = f(t)$  and  $x = g(t)$ , that is, if  $y$  and  $x$  are related parametrically, then

$$\frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)}$$

### Example 6

If  $y = t^3$  and  $x = 2t^2$ , then

$$\frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{\frac{d}{dt}(t^3)}{\frac{d}{dt}(2t^2)} = \frac{3t^2}{4t} = \frac{3t}{4}$$

## Higher derivatives

The operation of differentiation of  $y = f(x)$  produces a new function  $y' = f'(x)$  called the first derivative. If we again differentiate  $y' = f'(x)$ , we produce another new function  $(y')' = y'' = f''(x)$  called the second derivative. If we continue this process, we have

$$y = f(x)$$

$$y' = f'(x) = \frac{d}{dx}(f(x))$$

$$y'' = f''(x) = \frac{d^2}{dx^2}(f(x))$$

$$y''' = f'''(x) = \frac{d^3}{dx^3}(f(x))$$

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$$y^{(n)} = f^{(n)}(x) = \frac{d^n}{dx^n}(f(x))$$

### Example 7

If  $y = x^5$  then,

$$\frac{dy}{dx} = 5x^4$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(5x^4) = 20x^3$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}(20x^3) = 60x^2$$

$$\frac{d^4y}{dx^4} = \frac{d}{dx}\left(\frac{d^3y}{dx^3}\right) = \frac{d}{dx}(60x^2) = 120x$$

$$\frac{d^5y}{dx^5} = \frac{d}{dx}\left(\frac{d^4y}{dx^4}\right) = \frac{d}{dx}(120x) = 120$$

$$\frac{d^6y}{dx^6} = \frac{d}{dx}\left(\frac{d^5y}{dx^5}\right) = \frac{d}{dx}(120) = 0$$

**Partial differentiation**

A partial derivative is the derivative with respect to one variable of a multivariable function, assuming all other variables to be constants. For example if  $y = f(x,y)$ , is a function depending on two variables 'x' and 'y', then the partial derivative of 'f' with respect to 'x' is obtained by assuming the variable 'y' to be a constant and taking the derivative of 'f' with respect to 'x'. This is represented as  $\frac{\partial}{\partial x} f(x,y)$ .

**Example 8**

If  $y = x \sin(t)$ , then

$$\frac{\partial y}{\partial x} = \frac{\partial}{\partial x} (x \sin(t)) = \sin t \frac{\partial}{\partial x} (x) = \sin t$$

In this example, since the partial derivative with respect to the variable 'x' is required, the variable 't' is assumed to be a constant and the derivative with respect to 'x' is obtained by following the general rules of differentiation.

**Example 9**

If  $z = x^2 y^3$ , then

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} (x^2 y^3) = x^2 \frac{\partial}{\partial y} (y^3) = x^2 (3y^2) = 3x^2 y^2$$

**General rules of partial differentiation**

- If the function 'z' is dependent on two variables 'x' and 'y', i.e., if  $z = f(x,y)$ , then

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right)$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right)$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right)$$

- If  $z = f(x,y)$  and  $x = h(t)$ ,  $y = g(t)$ , then the total derivative of 'z' with respect to 't' is given by

$$\frac{dz}{dt} = \frac{d}{dt}(f(x, y)) = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

**Example 10**

If  $z = xy$  and  $x = \cos(t)$ ,  $y = \sin(t)$ , then

$$\begin{aligned} \frac{dz}{dt} &= \frac{d}{dt}(z) = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = \frac{\partial}{\partial x}(xy) \frac{d}{dt}(\cos t) + \frac{\partial}{\partial y}(xy) \frac{d}{dt}(\sin t) \\ &= y(-\sin t) + x(\cos t) = -y \sin t + x \cos t \end{aligned}$$

**Section 2: Integration****Introduction**

The basic principle of integration is to reverse differentiation. An integral is sometimes referred to as *antiderivative*.

**Definition:** Any function  $F$  is said to be an antiderivative of another function, ' $f$ ' if and only if it satisfies the following relation:

$$F' = f$$

where

$$F' = \text{derivative of } F$$

Note that, the definition does not say *the antiderivative*, it says *an antiderivative*. This is because for any given function ' $f$ ', if there are any antiderivatives, then there are infinitely many antiderivatives. This is further explained with the help of the following example.

**Example 11**

For the function  $f(x) = 3x^2$ , the functions  $F(x) = x^3$ ,  $G(x) = x^3 + 15$ , and  $H(x) = x^3 - 38$  are all antiderivatives of  $f$ . In fact, in order to be an antiderivative of  $f$ , all that is required is that the function be of the form  $K(x) = x^3 + C$ , where  $C$  is any real number. This is so because the derivative of a constant function is always zero, so the differentiation process eliminates the ' $C$ '.

$$F'(x) = 3x^2 = f(x)$$

$$G'(x) = 3x^2 = f(x)$$

$$H'(x) = 3x^2 = f(x)$$

Although integration has been introduced as an antiderivative, the symbol for integration is ' $\int$ '. So to integrate a function  $f(x)$ , you write

$$\int f(x) dx$$

It is very essential to include the ' $dx$ ' as this tells someone the variable of integration.

**Definition:** The expression  $\int f(x) dx = F(x) + C$ , where  $C$  is any real number, means that  $F(x)$  is an antiderivative of  $f(x)$ . This expression represents the *indefinite integral of  $f(x)$* .

The 'C' in the above definition is called the *constant of integration* and is absolutely vital to indefinite integration. If it is omitted, then you only find one of the infinitely many antiderivatives.

### Some properties of indefinite integrals

- $\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$
- $\int cf(x)dx = c \int f(x)dx$  where c is any real constant

### Some of the common integration formulae

$$\int 0dx = c$$

$$\int xdx = \frac{x^2}{2} + c$$

$$\int x^2 dx = \frac{x^3}{3} + c$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\int \frac{1}{x} dx = \ln x + c$$

$$\int \sin x dx = -\cos x + c$$

$$\int \cos x dx = \sin x + c$$

The constant 'c' can be found if some more information about the antiderivative is given. This is explained with the aid of the following example.

### Example 12

Find the function whose derivative is  $f(x) = 3x^2 + 1$  and which passes through the point (2,5).

In other words, the function 'f' has to be integrated with respect to 'x'. So

$$F(x) = \int f(x)dx = \int (3x^2 + 1)dx = \int 3x^2 dx + \int 1dx$$

$$\Rightarrow F(x) = x^3 + x + c$$

It is given that, the function F(x) passes through the point (2,5), i.e., when  $x = 2$ ,  $F(2) = 5$ .



$$F(x) = F(2) = (2)^3 + 2 + c = 10 + c$$

$$\Rightarrow 5 = 10 + c$$

$$\Rightarrow c = -5$$

Therefore the function  $F(x)$  is given by

$$F(x) = x^3 + x - 5.$$

### Definite integration

This is very much similar to the indefinite integration, except that the limits of integration are specified. Since the limits are specified, there is no need to put the constant of integration. In other words

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

### Example 13

Evaluate the following integral

$$\int_1^2 xdx.$$

$$\int_1^2 xdx = \frac{x^2}{2}\Big|_1^2 = \frac{(2)^2}{2} - \frac{(1)^2}{2} = 2 - \frac{1}{2} = \frac{3}{2}$$

### Example 14

$$I = \int_0^5 (2x + 3)dx = \int_0^5 2xdx + \int_0^5 3dx = x^2\Big|_0^5 + 3x\Big|_0^5$$

$$\Rightarrow I = \{(5)^2 - (0)^2\} + \{3(5) - 3(0)\}$$

$$\Rightarrow I = \{25 - 0\} + \{15 - 0\}$$

$$\Rightarrow I = 40$$

**Integration by parts***Formulae*

If  $u = f(x)$  and  $v = g(x)$  and if  $f'$  and  $g'$  are continuous, then

$$\int u dv = uv - \int v du \quad (1)$$

For definite integrals

$$\int_a^b f(x)g'(x)dx = [f(x)g(x)]_a^b - \int_a^b g(x)f'(x)dx$$

The following example illustrates how the integration-by-parts formula works.

**Example 15**

Evaluate the integral

$$I = \int xe^x dx$$

$$\text{Let } u = x \text{ and } dv = e^x dx \quad (2)$$

So

$$du = 1 \text{ and } v = e^x \quad (3)$$

Since

$$v = \int dv = \int e^x dx = e^x$$

Thus substituting equations (2) and (3) in equation (1), we have

$$\int xe^x dx = xe^x - \int e^x(1)dx + c$$

$$\Rightarrow I = xe^x - e^x + c$$

**Example 16**

Find  $\int e^x \cos x dx$

Let

$$u = e^x, \quad dv = \cos x dx$$

so

$$du = e^x, \quad v = \sin x$$

Substituting the above relation in equation (1), we get

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx + c. \quad (4)$$

Consider again the integral  $\int e^x \sin x dx$ . Let

$$u = e^x, \quad dv = \sin x dx$$

so

$$du = e^x, \quad v = -\cos x$$

Then we have

$$\begin{aligned} \int e^x \sin x dx &= -e^x \cos x - \int e^x (-\cos x) dx + c \\ \Rightarrow \int e^x \sin x dx &= -e^x \cos x + \int e^x \cos x dx + c \end{aligned} \quad (5)$$

Substituting the second equation of equation (5) in equation (4), we get

$$\begin{aligned} \int e^x \cos x dx &= e^x \sin x - (-e^x \cos x + \int e^x \cos x dx) + c \\ \Rightarrow 2 \int e^x \cos x dx &= e^x (\sin x + \cos x) + c \\ \Rightarrow \int e^x \cos x dx &= \frac{1}{2} e^x (\sin x + \cos x) + c \end{aligned}$$

**Leibniz rule**

The fundamental theorem of integral calculus states that if  $f$  is a continuous function on the closed interval  $[a, b]$ , then

$$\frac{d}{dt} \int_a^t f(x) dx = f(t)$$

for  $a \leq t \leq b$

The Leibniz rule can be derived from the above fundamental theorem of calculus. If

$$I(t) = \int_{a(t)}^{b(t)} f(x, t) dx \quad (6)$$

The final relation, which outlines the Leibniz rule is given by

$$\frac{d}{dt}(I(t)) = \frac{d}{dt} \int_{a(t)}^{b(t)} f(x, t) dx = \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(x, t) dx + \frac{d}{dt}(b(t))f(b(t), t) - \frac{d}{dt}(a(t))f(a(t), t) \quad (7)$$

This rule is useful when one needs to find the derivative of an integral without actually evaluating the integral. The rule is further explained with the aid of the following example.

**Example 17**

Given

$$I(t) = \int_{-t}^{t^2} \cos(tx^2) dx$$

Evaluate  $\frac{d}{dt}(I(t))$ .

The Leibniz rule can be applied to evaluate the derivative without actually evaluating the integral. Comparing the above integral with equation (6), we have

$$f(x, t) = \cos(tx^2)$$

$$a(t) = -t$$

$$b(t) = t^2$$

$$\frac{\partial}{\partial t}(f(x, t)) = \frac{\partial}{\partial t}(\cos(tx^2)) = -x^2 \sin(tx^2)$$

Substituting the above relations in equation (7), we get

$$\begin{aligned}\frac{d}{dt}(I(t)) &= -\int_{-t}^{t^2} x^2 \sin(tx^2) dx + (2t)(\cos[t(t^2)^2]) - (-1)(\cos[t(-t)^2]) \\ \Rightarrow \frac{d}{dt}(I(t)) &= -\int_{-t}^{t^2} x^2 \sin(tx^2) dx + 2t \cos(t^5) + \cos(t^3).\end{aligned}$$

### L'Hospitals rule

L'Hospitals rule gives a way to evaluate limits of the form  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$  if the numerator and denominator both tend to zero, that is,  $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$ . The rule states that if the numerator and the denominator both tend to zero, then the limit of the quotient  $\frac{f(x)}{g(x)}$  is equal to the limit of the quotient of the derivatives  $\frac{f'(x)}{g'(x)}$ . In other words

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

### Example 18

Evaluate  $\lim_{x \rightarrow 0} \frac{\sin x}{x}$ .

It can be seen that both the numerator and the denominator tend to zero, as 'x' tends to zero. Therefore applying the L'Hospitals rule, we have

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \frac{\left(\frac{d}{dx}(\sin x)\right)}{\left(\frac{d}{dx}(x)\right)} = \lim_{x \rightarrow 0} \frac{\cos x}{1} = \frac{1}{1} = 1$$

**Assignment**

1) Calculate the derivatives of the following functions with respect to 'x'.

a)  $y = 2x^3 - 4$

b)  $y = \frac{x+5}{2x+3}$

c)  $y = x^2 \sin x$

2) Calculate the partial derivatives,  $\frac{\partial y}{\partial x}$  and  $\frac{\partial y}{\partial t}$  for the following functions.

a)  $y = x^2 \sin^2 t$

b)  $y = x^3 t^6 + \tan x \sin t \cos t$

3) Calculate the following indefinite integrals analytically and verify the result Using MATLAB.

a)  $\int \cos^2 x dx$

b)  $\int x^2 \sin x dx$

c)  $\int x \tan x dx$

4) Evaluate the following definite integrals and verify the result using MATLAB.

a)  $\int_1^2 (x^4 + 7x^3 + 5x^2 + 6x + 9) dx$

b)  $\int_0^{\pi/2} \sin x dx$

5) Evaluate the derivative with respect to time of the following integral by means of Leibniz rule.

a)  $I(t) = \int_0^{t^2} \sin(tx^2) dx$

6) Evaluate the following limits:

$$a) \lim_{x \rightarrow 0} \frac{x^2 + 2x}{\sin x}$$

$$b) \lim_{x \rightarrow -1} \frac{x + 1}{x^2 - 1}$$

$$c) \lim_{x \rightarrow 0} \frac{\sin(x^3)}{\sin(x^2)}$$

**NOTE:** This handout is not a comprehensive tutorial for differentiation and integration. This just deals with the very basics of differentiation and integration. It is advisable always to go through some MATH book for various other techniques of performing differentiation and integration.