HADNOUT E.13 - EXAMPLES ON TRANSFER FUNCTIONS, POLES AND ZEROS

Example 1

Determine the transfer function of the mass-spring-damper system.

The governing differential equation of a mass-spring-damper system is given by

$m\ddot{x} + c\dot{x} + kx = F$.

Taking the Laplace transforms of the above equation (assuming zero initial conditions), we have

$$ms^2X(s) + csX(s) + kX(s) = F(s),$$
$$\Rightarrow \frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k}. \tag{1}$$

Equation (1) represents the transfer function of the mass-spring-damper system.

Example 2

Consider the system given by the differential equation

$\ddot{y} + 4\dot{y} + 3y = 2r(t)$, where r(t) is the input to the system. Assume zero initial conditions.

The Laplace transforms yields,

$$s^2Y(s) + 4sY(s) + 3Y(s) = 2R(s),$$
$$\Rightarrow \frac{Y(s)}{R(s)} = \frac{2}{s^2 + 4s + 3}. \tag{2}$$

Equation (2) represents the transfer function of the system.

Example 3

Find the solution of the differential equation

$\ddot{y}(t) + y(t) = 0$, where $y(0) = \alpha$ and $\dot{y}(0) = \beta$.

The Laplace transform of the above differential equation gives,

$$s^2Y(s) - \alpha\, s - \beta\ + Y(s) = 0,$$
$$\Rightarrow Y(s) = \frac{\alpha\, s}{s^2 + 1} + \frac{\beta}{s^2 + 1}.$$

After looking up in the transform tables, the two terms in the right side of the above equation, we get

$$y(t) = \alpha \ \cos(t) + \beta \ \sin(t).$$

Example 4

Consider a RLC circuit. The governing differential equation is given by

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int idt = V. \qquad (3)$$

But,

$$i = \frac{dq}{dt}.$$

Therefore, equation (3) reduces to

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = V.$$

The Laplace transforms of the above equation yields

$$Ls^2Q(s) + RsQ(s) + \frac{1}{C}Q(s) = V(s),$$

$$\Rightarrow \frac{Q(s)}{V(s)} = \frac{1}{Ls^2 + Rs + \frac{1}{C}}.$$

The above equation represents the transfer function of a RLC circuit.

Example 5

Determine the poles and zeros of the system whose transfer function is given by

$$G(s) = \frac{2s+1}{s^2+3s+2}.$$

The zeros of the system can be obtained by equating the numerator of the transfer function to zero, i.e.,

$$2s + 1 = 0,$$
$$\Rightarrow s = -\frac{1}{2}.$$

The poles of the system can be obtained by equating the denominator of the transfer function to zero, i.e.,

$$s^2 + 3s + 2 = 0,$$
$$\Rightarrow (s+1)(s+2) = 0,$$
$$\Rightarrow s = -1, \quad s = -2.$$

Therefore s = -1 and s = -2 are the poles of the system and s = -1/2 is the zero of the system.

Example 6

Determine the poles and zeros of the system, whose transfer function is given by

$$H(s) = \frac{30(s-6)}{s(s^2 + 4s + 13)}.$$

The zeros of the system are given by

$$30(s-6) = 0,$$
$$\Rightarrow s = 6.$$

Therefore s = 6 is the zero of the system.

The poles of the system are given by

$$s(s^2 + 4s + 13) = 0,$$
$$\Rightarrow s = 0, \quad s^2 + 4s + 13 = 0,$$
$$\Rightarrow s = 0, \quad s = \frac{-4 \pm \sqrt{16 - 52}}{2},$$
$$\Rightarrow s = 0, \quad s = \frac{-4 + i6}{2}, \quad s = \frac{-4 - i6}{2},$$
$$\Rightarrow s = 0, \quad s = -2 + i3, \quad s = -2 - i3.$$

Therefore the poles of the system are s = 0, s = -2 + i3 and s = -2 – i3.

Example 7

Consider the mass-spring-damper system. The governing differential equation of motion for the system is given by

$$m\ddot{x} + c\dot{x} + kx = F. \tag{4}$$

Let the states of the system be defined as

$$\begin{aligned} x &= x_1, \\ \dot{x} &= x_2. \end{aligned} \tag{5}$$

From the above relation, it can be concluded that

$$\dot{x}_1 = x_2. \tag{6}$$

Substituting the relations given by equation (5) in equation (4), we get

$$m\ddot{x} + c\dot{x} + kx = F,$$

$$\Rightarrow m\dot{x}_2 + cx_2 + kx_1 = F,$$

$$\Rightarrow \dot{x}_2 = -\frac{k}{m}x_1 - \frac{c}{m}x_2 + \frac{1}{m}F. \tag{7}$$

Representing equations (6) and (7) in matrix format, we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} F. \tag{8}$$

If the output of the system is the velocity of the mass, then writing the output relation in matrix format, we get

$$y = \dot{x} = x_2,$$

$$\Rightarrow y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{9}$$

Equations (8) and (9) represent the state-space representation of the mass-spring-damper system.

Obtaining the transfer function from the state-space representation

Given the '**A**', '**B**', '**C**' and '**D**' matrices of the state-space equations, the transfer function of the system is given by

$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}.$

From equation (8), the transfer function can be written as

$$G(s) = \begin{bmatrix} 0 & 1 \end{bmatrix} \left[s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \right]^{-1} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix},$$

$$\Rightarrow G(s) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s & -1 \\ \frac{k}{m} & s + \frac{c}{m} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix},$$

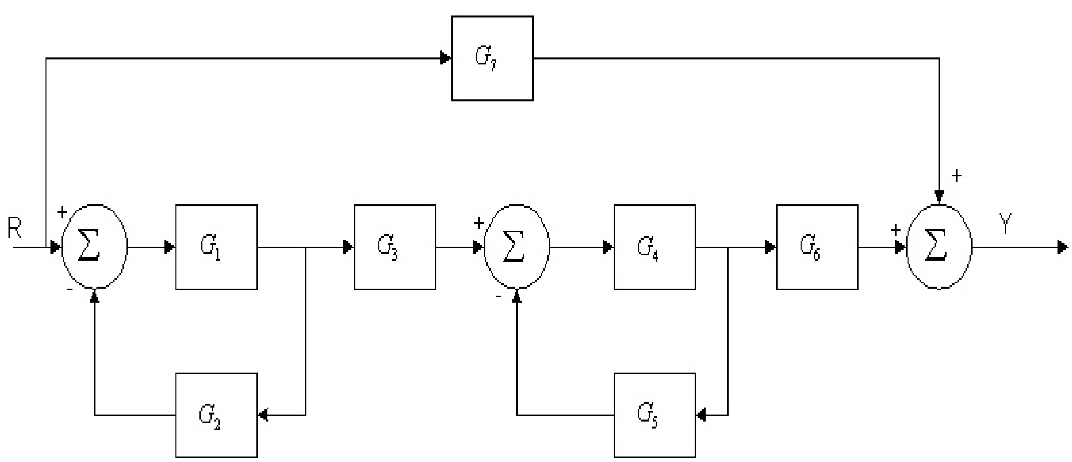
$$\Rightarrow G(s) = \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{1}{s\left(s + \frac{c}{m}\right) + \frac{k}{m}} \begin{bmatrix} s + \frac{c}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} = \frac{1}{s\left(s + \frac{c}{m}\right) + \frac{k}{m}} \begin{bmatrix} -\frac{k}{m} & s \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix},$$

$$\Rightarrow G(s) = \frac{m}{ms^2 + cs + k}\left(\frac{s}{m}\right) = \frac{s}{s^2 + cs + k}. \qquad (10)$$

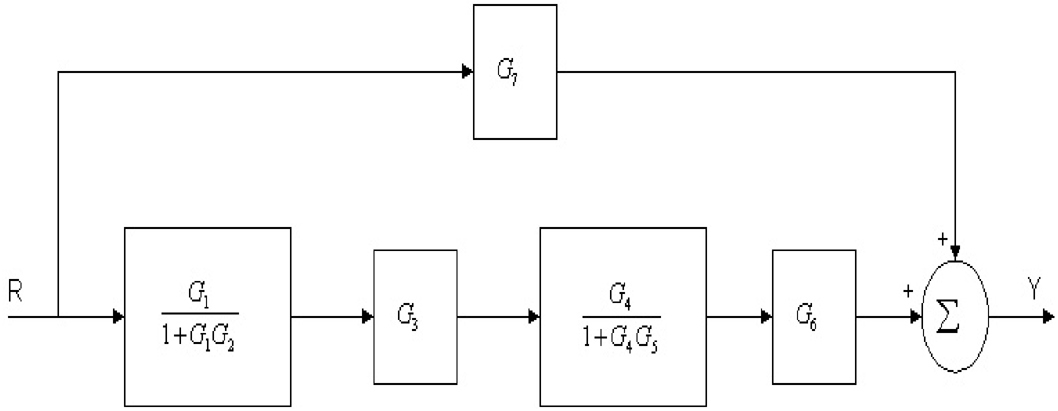
Equation (10) represents the transfer function of the system, wherein the input to the system is the force applied to the system and the output of the system is the velocity of the mass.

Example 8

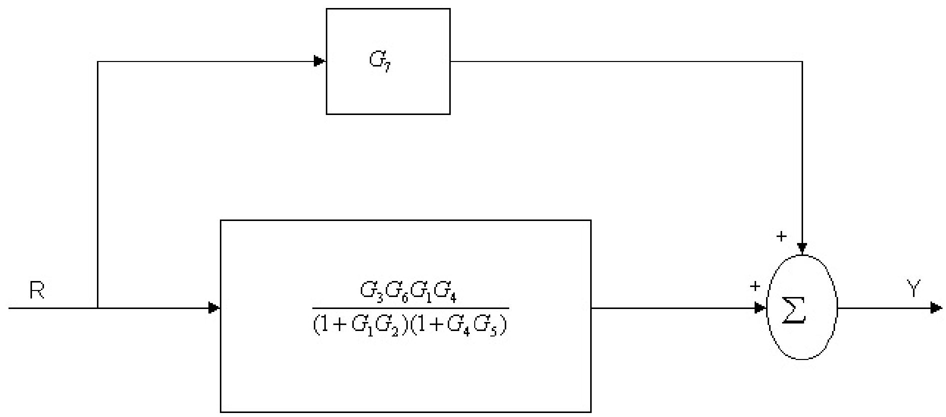
Find the transfer function for the block diagram shown below.



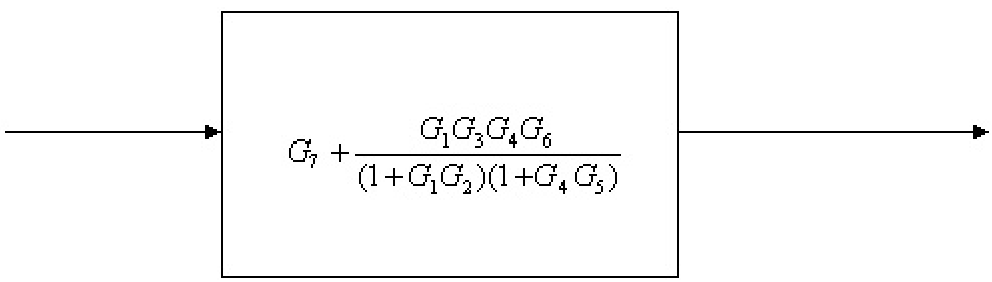
Sol: Note that there are two negative feedback loop and one positive feed-forward loop. Reducing the negative feedback loops, we get



The above diagram can be further reduced to



Feed-forward loop when reduced yields



Assignment

1) Determine the transfer functions for the following systems, whose differential equations are given by

$$J\ddot{\theta} + B\dot{\theta} = K_T i_a,$$
$$L\frac{di_a}{dt} + Ri_a = v_a - K_e\dot{\theta}.$$

The input to the system is the voltage, 'v_a', whereas the output is the angle 'θ'.

2) Determine the poles and zeros of the system whose transfer functions are given by

$$a) \quad G(s) = \frac{(s+10)(s^2+s+25)}{s^2(s+3)(s^2+s+36)}$$
$$b) \quad H(s) = \frac{(s^2+5s+6)}{(s+8)(s^2+2s+32)}.$$

3) Obtain the state-space representation of the system whose differential equation is given by the equation given in example 2.