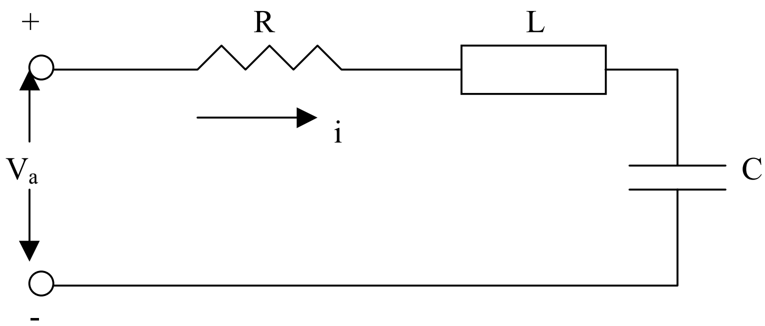
HANDOUT E.8 - EXAMPLES ON MODELLING OF ELECTRICAL, ELECTROMECHANICAL SYSTEMS

Note that the time dependence of variables is ignored for all manipulations.

Example 1: An electrical circuit

Consider the circuit shown below.



Writing the Loop equation for the above circuit, we have

$$V_a - Ri - L\frac{di}{dt} - \frac{1}{C}\int idt = 0. \quad (1)$$

We know that

$$i = \frac{dq}{dt}.$$

Therefore substituting the above relation in equation (1), we have

$$V_a = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C}. \quad (2)$$

Equation (2) represents the governing differential equation of the circuit shown above.

State-space representation

Let the states of the system be defined as

$$q = x_1, \quad \dot{q} = x_2. \quad (3)$$

From the above relations, we get

$$\dot{x}_1 = x_2. \quad (4)$$

Substituting the relations given by equation (3) in equation (2), we get

$$V_a = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C},$$

$$V_a = L\dot{x}_2 + Rx_2 + \frac{1}{C}x_1,$$

$$\Rightarrow \dot{x}_2 = -\frac{1}{LC}x_1 - \frac{R}{L}x_2 + \frac{1}{L}V_a. \tag{5}$$

Rewriting equations (4) and (5) in matrix format, we have

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{1}{LC} & -\frac{R}{L}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{L}\end{bmatrix}V_a. \tag{6}$$

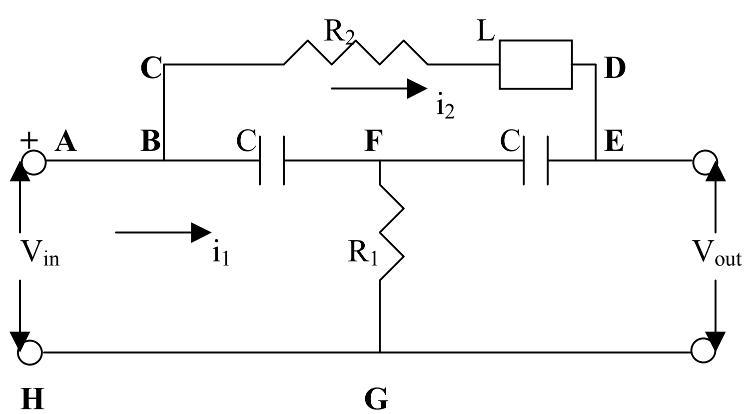
If the output of the system is the voltage drop across the capacitor, then the output equation can be written as

$$y = \frac{q}{C} = \frac{1}{C}x_1,$$

$$\Rightarrow y = \begin{bmatrix}\frac{1}{C} & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}. \tag{7}$$

Equations (6) and (7) represent the state-space form of the circuit.

Example 2: An electrical circuit



Note that,

$$i_1 = \frac{dq_1}{dt}, \quad i_2 = \frac{dq_2}{dt}. \tag{8}$$

Writing the loop closure equation for the loop ABFGHA, we have

$$V_{in} - \frac{1}{C}\int (i_1 - i_2)dt - R_1 i_1 = 0. \tag{9}$$

Substituting equation (8) in equation (9), we have

$$V_{in} - \frac{q_1 - q_2}{C} - R_1 \frac{dq_1}{dt} = 0. \tag{10}$$

Writing the loop closure equation for the loop BCDEFB, we get

$$\begin{aligned} &R_2 i_2 + L\frac{di_2}{dt} + \frac{1}{C}\int i_2 dt + \frac{1}{C}\int (i_2 - i_1)dt = 0, \\ &\Rightarrow L\frac{d^2 q_2}{dt^2} + R_2 \frac{dq_2}{dt} + \frac{2q_2}{C} - \frac{q_1}{C} = 0. \end{aligned} \tag{11}$$

Equations (10) and (11) represent the governing differential equations for the circuit shown.

State-space representation

Let the states of the system be defined as

$$\begin{aligned} q_1 &= x_1, \\ q_2 &= x_2, \\ \dot{q}_2 &= x_3. \end{aligned} \tag{12}$$

From the above relations, the following equation can be deduced.

$$\dot{x}_2 = x_3. \tag{13}$$

Substituting the relations given by equation (12) in equation (10), we have

$$\begin{aligned} &V_{in} - \frac{q_1 - q_2}{C} - R_1 \frac{dq_1}{dt} = 0, \\ &\Rightarrow V_{in} - \frac{1}{C}x_1 + \frac{1}{C}x_2 - R_1 \dot{x}_1 = 0, \\ &\Rightarrow \dot{x}_1 = -\frac{1}{R_1 C}x_1 + \frac{1}{R_1 C}x_2 + \frac{1}{R_1}V_{in}. \end{aligned} \tag{14}$$

Similarly substituting the relations given by equation (12) in equation (11), we get

$$L\frac{d^2q_2}{dt^2}+R_2\frac{dq_2}{dt}+\frac{2q_2}{C}-\frac{q_1}{C}=0,$$

$$\Rightarrow L\dot{x}_3+R_2x_3+\frac{2}{C}x_2-\frac{1}{C}x_1=0,$$

$$\Rightarrow \dot{x}_3=\frac{1}{LC}x_1-\frac{2}{LC}x_2-\frac{R_2}{L}x_3. \qquad (15)$$

Rewriting equations (13), (14) and (15) in matrix format, we have

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}-\frac{1}{R_1C} & \frac{1}{R_1C} & 0\\ 0 & 0 & 1\\ \frac{1}{LC} & -\frac{2}{LC} & -\frac{R_2}{L}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}\frac{1}{R_1}\\ 0\\ 0\end{bmatrix}V_{in}. \qquad (16)$$

From the circuit it can be seen that the output is given by

$$V_{out}=\frac{1}{C}\int i_2dt+R_1i_1,$$

$$\Rightarrow V_{out}=\frac{1}{C}q_2+R_1\frac{dq_1}{dt},$$

Substituting the value of $R_1\frac{dq_1}{dt}$ from equation (10) in the above relation, we get

$$V_{out}=\frac{1}{C}q_2+V_{in}-\frac{q_1-q_2}{C},$$

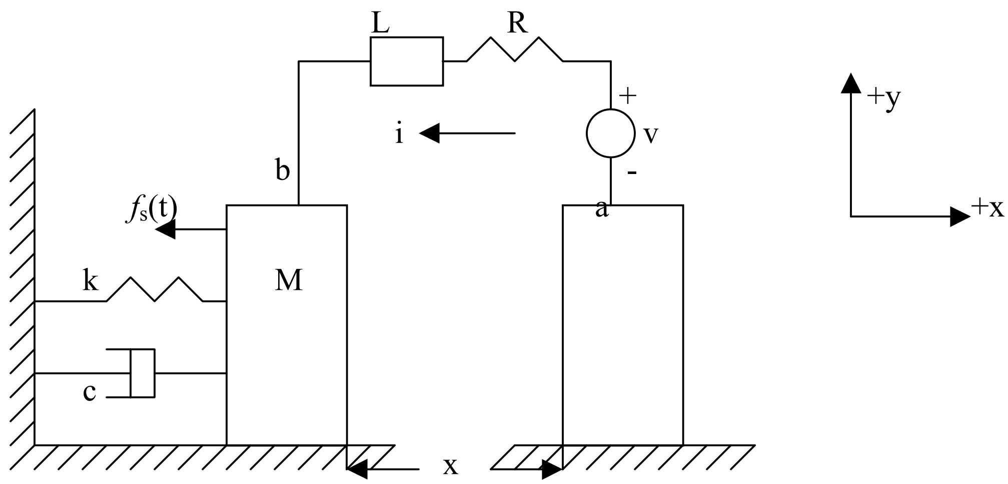
$$\Rightarrow V_{out}=\frac{2}{C}x_2-\frac{1}{C}x_1+V_{in},$$

$$\Rightarrow Y=\begin{bmatrix}-\frac{1}{C} & \frac{2}{C} & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+[1]V_{in}. \qquad (17)$$

Therefore equations (16) and (17) represent the state-space form of the above circuit.

Example 3: An electromechanical system

Consider the following electromechanical system shown.



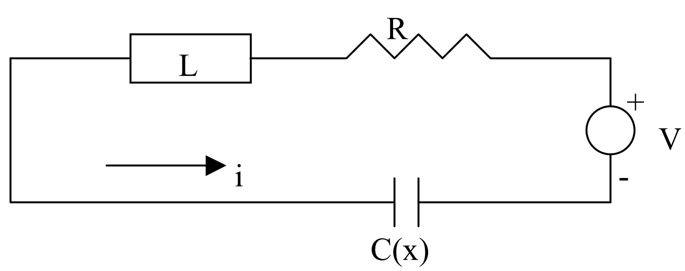
The electromechanical system shown above represents a simplified model of a capacitor microphone. The system consists a parallel plate capacitor connected into an electric circuit. Capacitor plate 'a' is rigidly fastened to the microphone frame. Sound waves pass through the mouthpiece and exert a force f_s(t) on plate 'b', which has mass, 'M' and is connected to the frame by a set of springs and dampers. The capacitance C is a function of the distance x between the plates. The electric field in turn produces a force f_e on the movable plate that opposes its motion.

Kinematics stage

Let the movable plate 'b' move a distance 'x' units. Then the velocity and the acceleration of the plate is given by $\dot{x}$, $\ddot{x}$ respectively. It can be seen from the figure that the current 'i' flows through the electric circuit in the counter clockwise direction.

Kinetics stage

First let us consider the electric circuit.



Writing the loop closure equation for the above circuit, we get

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int idt = V. \quad (18)$$

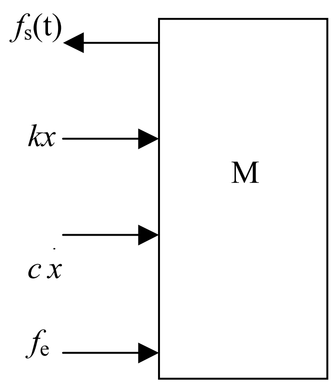
Since,

$$i = \frac{dq}{dt},$$

equation (18) reduces to

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V. \quad (19)$$

Free body diagram of the movable plate of mass M



Writing the Newton's second law of motion, we have

$$\sum F_x = ma,$$

$$\Rightarrow kx + c\dot{x} + f_e - f_s(t) = -m\ddot{x},$$

$$\Rightarrow m\ddot{x} + c\dot{x} + kx + f_e = f_s(t). \quad (20)$$

Equations (19) and (20) represent the governing differential equations of motion for the electromechanical system considered. The force f_e is defined as

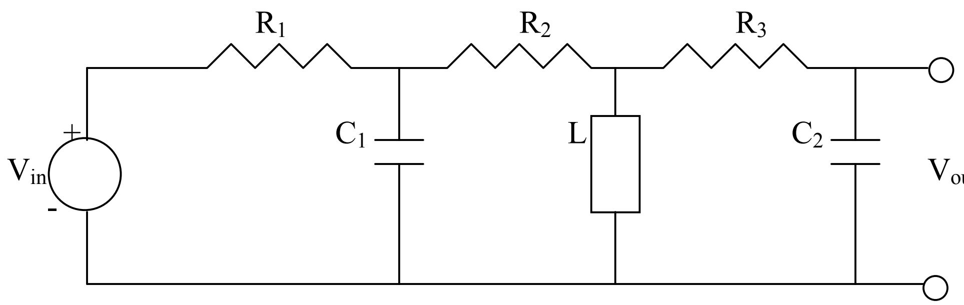
$$f_e = \frac{q^2}{2\varepsilon A},$$

where A is the surface area of the plates and ε is the dielectric constant of the material between the plates.

Since f_e is non-linear in nature, equation (20) is a non-linear equation. The linearization of a non-linear equation is explained in the handout on linearization. Once the equation is linearized, then it can be represented in the state-space form. This section is dealt with, in detail in a later handout.

Assignment

1) Find the differential equations for the circuit shown below and put them in state-variable form.



2) Consider the schematics of an electromechanical shaker as shown below. This system consists of a table of mass M, and a coil whose mass is m. A permanent magnet rigidly attached to the ground provides a steady magnetic field, i.e., the motion of the coil through the magnetic field induces a voltage in the coil that is proportional to its velocity. The passage of current through the coil causes it to experience a magnetic force proportional to the current. Derive the equations governing the dynamics of this system.