

**Texas A & M University**  
**Department of Mechanical Engineering**  
**MEEN 364 Dynamic Systems and Controls**  
Dr. Alexander G. Parlos

**Lecture 3: Modeling of Rotational Mechanical Systems**

The objective of this lecture is to review the basic building blocks of lumped parameter rotational mechanical systems and to build the foundations that will enable you to model more complex dynamic systems consisting of translation and rotational elements. Corresponding to the translational elements mass, spring and damper, are the rotational inertia, rotational spring and rotational damper, respectively. We now discuss these elements briefly.

**Rotational Inertia Elements:**

An ideal rotational inertia is shown in the Figure 1. Motion is considered with respect to a non-accelerating rotating reference frame, usually a fixed point on earth or another non-accelerating rotating object.

The equation of motion for an ideal inertia,  $J(t)$ , is based on Newton's second law for rotational motion which expresses the conservation of angular momentum, as follows:

$$\frac{d(J(t)\Omega_{1g}(t))}{dt} = \sum_{i=1}^n T_i(t), \quad (1)$$

where if we assume that the inertia is a constant,  $J$ , we can rewrite this equation as

$$J\left(\frac{d\Omega_{1g}(t)}{dt}\right) = \sum_{i=1}^n T_i(t) = T_J, \quad (2)$$

where  $\Omega_{1g}(t)$  is the angular velocity of the inertia relative to the ground reference (g) and  $T_J$  is the *net* torque (or twisting moment) acting on the

inertia.

Because of the non-accelerating nature of the rotating reference frame

$$\frac{d\Omega_{1g}(t)}{dt} = \frac{d\Omega_1(t)}{dt}, \quad (3)$$

and

$$\Omega_{1g}(t) = \frac{d\theta_1(t)}{dt}, \quad (4)$$

resulting in the following equation of motion for the ideal inertia,  $J$

$$J\left(\frac{d^2\theta_1(t)}{dt^2}\right) = T_J(t). \quad (5)$$

Furthermore, the action of the applied torque represents work being done on the inertia as it accelerates, increasing its kinetic energy. The rate at which energy is stored in the system is equal to the rate at which work is expended on it. Using the first law of thermodynamics (or the law of conservation of energy) we have

$$\frac{d\mathcal{E}_K(t)}{dt} = T_J(t)\Omega_{1g}(t). \quad (6)$$

To find the energy of the system we need to integrate over a period  $[0, t]$ , as follows:

$$\int_{\mathcal{E}_K(0)}^{\mathcal{E}_K(t)} d\mathcal{E}_K = \int_0^t T_J(t)\Omega_{1g}(t)dt = \int_0^t J\Omega_{1g}(t)\left(\frac{d\Omega_{1g}(t)}{dt}\right)dt = J \int_{\Omega_{1g}(0)}^{\Omega_{1g}(t)} \Omega_{1g}(t)d\Omega_{1g}, \quad (7)$$

or

$$\mathcal{E}_K(t) = \mathcal{E}_K(0) + \left(\frac{J}{2}\right)\Omega_{1g}^2(t). \quad (8)$$

This is the well-know formula for the kinetic energy of a rotational inertial element.

Equation (5) indicates that because of the integrations involved, it takes some time for the rotating object to build-up angular velocity and angular displacement. As such, it would not be realistic to attempt to apply a step change in *angular velocity* (or *angular displacement*) of the inertia. This would require an infinite torque and an infinite source of power!

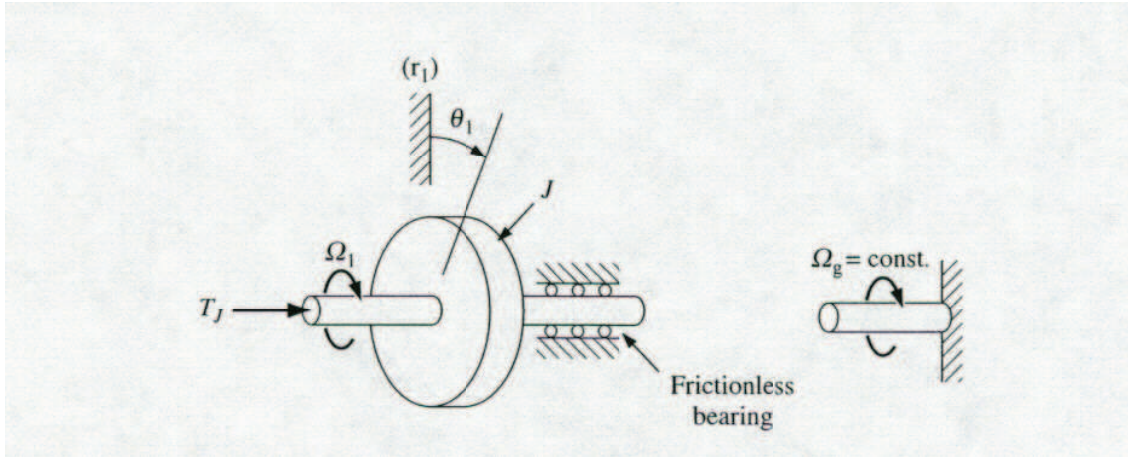


Figure 1: Free-body diagram of an ideal rotational inertia.

### Rotational Stiffness Elements (or Springs):

An ideal rotational spring, stores potential energy as it is twisted, i.e. wound up. This is depicted in Figure 2. The figure shows a spring in its relaxed state of no torques acting,  $T_K = 0$ , and with the torque,  $T_K$ , acting at both ends, in free-body diagram fashion. Because an ideal rotational spring has no mass, the torque transmitted by it is undiminished during rotational acceleration. Therefore, the torques acting on its ends must be equal and opposite (Newton's third law of motion). The elemental equation for such a rotational spring derives from Hooke's law, namely

$$T_K(t) = K[\theta_1(t) - \theta_2(t)], \quad (9)$$

where  $\theta_1(t)$  and  $\theta_2(t)$  are the angular displacements of the ends from their local references  $r_1$  and  $r_2$ .

In derivative form, this equation becomes

$$\frac{T_K(t)}{dt} = K[\Omega_{1g}(t) - \Omega_{2g}(t)], \quad (10)$$

where  $\Omega_{1g}(t)$  and  $\Omega_{2g}(t)$  are the angular velocities of the ends relative to the non-accelerating reference  $g$ . In each case, the sign convention employed for

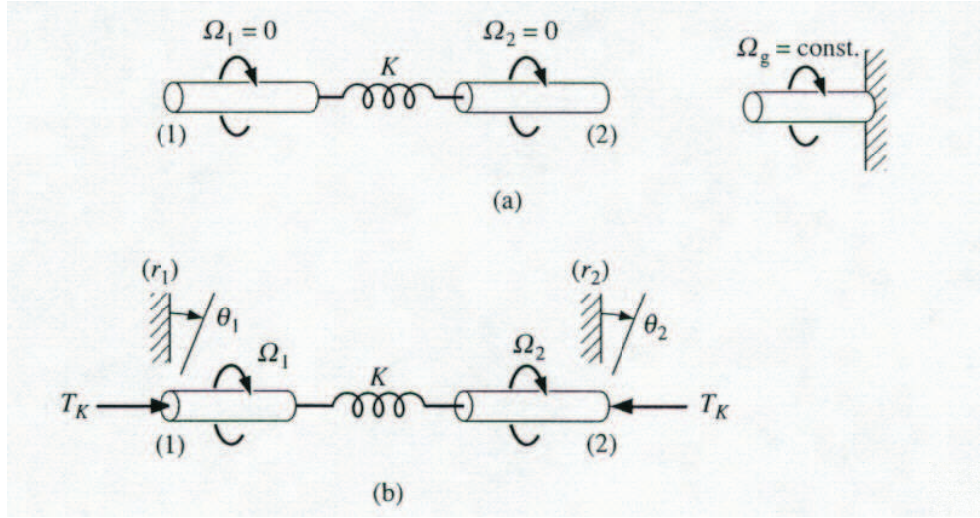


Figure 2: Free-body diagram of an ideal rotational spring.

motion is clockwise positive when viewed from the left.

In this development we have assumed that the rotational spring has a constant stiffness  $K$ . If this is not the case, we can write the general form of equation (9) as

$$T_{NLS}(t) = f_{NL}(\theta_1(t) - \theta_2(t)), \quad (11)$$

where  $NL$  stands for a non-linear rotational spring. Simplification of this equation, via linearization, to a rotational spring with an equivalent stiffness near an operating point will be discussed in future lectures.

Similar to the case of an ideal inertia, we can investigate the energy point-of-view of an ideal rotational spring. The conservation of energy for an ideal rotational spring can be expressed as

$$\frac{d\mathcal{E}_P(t)}{dt} = T_K(t)\Omega_{1g}(t). \quad (12)$$

To find the energy of the system we need to integrate, as follows:

$$\int_{\mathcal{E}_P(0)}^{\mathcal{E}_P(t)} d\mathcal{E}_P = \int_0^t T_K(t)\Omega_{1g}(t)dt = \left(\frac{1}{K}\right) \int_0^t T_K(t)\left(\frac{dT_K(t)}{dt}\right)dt = \left(\frac{1}{K}\right) \int_{T_K(0)}^{T_K(t)} T_K(t)dT_K, \quad (13)$$

or

$$\mathcal{E}_P(t) = \left(\frac{1}{2K}\right)T_K^2(t). \quad (14)$$

This is the formula for the potential energy of an ideal rotational spring.

As in the case of the ideal inertia, it would not be realistic to attempt to apply a step change in *torque* to a spring. Such a torque would have to rotate infinitely fast to twist the spring suddenly, which would require an infinite source of power!

### **Rotational Damping Elements (or Dampers):**

Just as friction between moving parts of a translational system give rise to translational damping, friction between rotating parts in a rotational system is the source of rotational damping. An ideal rotational damper is shown in Figure 3. Because an ideal rotational damper contains no mass, the torque transmitted through it is undiminished during rotational acceleration. Therefore, the torques acting at its ends must always be equal and opposite. The basic equation of an ideal rotational damper is

$$T_B(t) = B(\Omega_{1g}(t) - \Omega_{2g}(t)), \quad (15)$$

where  $T_B(t)$  is the torque transmitted by the damper. With a rotational damper there is no storage of retrievable mechanical work, as the work being done by an applied torque becomes dissipated as thermal internal energy. The relationship between the torque and angular velocity is instantaneous. Thus, it is realistic to apply step changes of either *torque* or *angular velocity* to such an element.

An ideal rotating damper arises from viscous friction between well-lubricated rotating mechanical parts of a system. This is the only form of damping that is linear. Non-ideal forms of damping are very common in practice. However, as in the case of translational motion, non-ideal rotational damping is

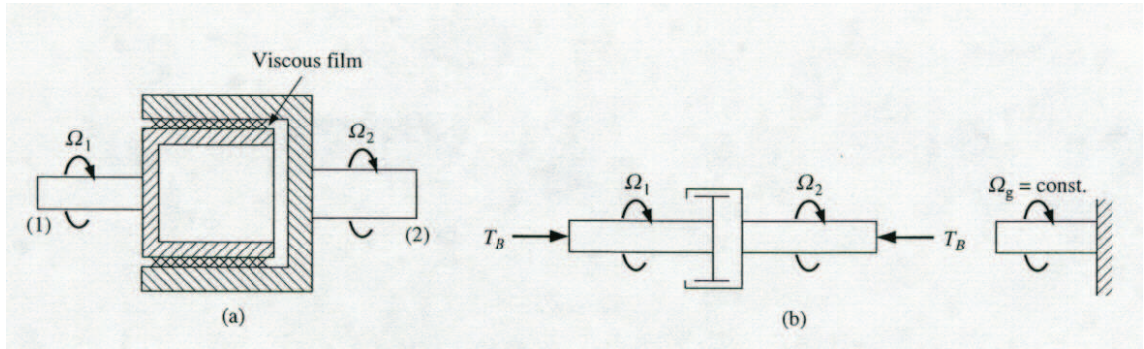


Figure 3: Free-body diagram of an ideal rotational damper.

characterized by non-linearities, such as dry (Coulomb) friction, aerodynamic damping, structural damping. These forms of damping can be linearized for simplification when no discontinuities exist in the torque-angular velocity characteristics of the damper, at the expense of accuracy of the analysis. Non-ideal rotational damping is found, for example, in poorly lubricated metal-metal contact rotating surfaces.

In general, for a nonlinear rotational damper we can write

$$T_{NLD}(t) = f_{NL}(\Omega_{1g}(t), \Omega_{2g}(t)), \quad (16)$$

where  $NL$  stands for a non-linear rotational damper. Simplification of this equation, via linearization, to a rotational damper with equivalent linear damping near an operating point will be discussed in future lectures.

### Example: A Simplified Ship Propulsion System:

The power transmission system from the diesel engine to propeller for a ship is shown in simplified form in Figure 4. The role of the fluid coupling is to transmit the main flow of power from the engine to the propeller shaft without allowing excessive vibration, which would otherwise be caused by pulsations of engine torque resulting from the cyclic firing of its cylinders.

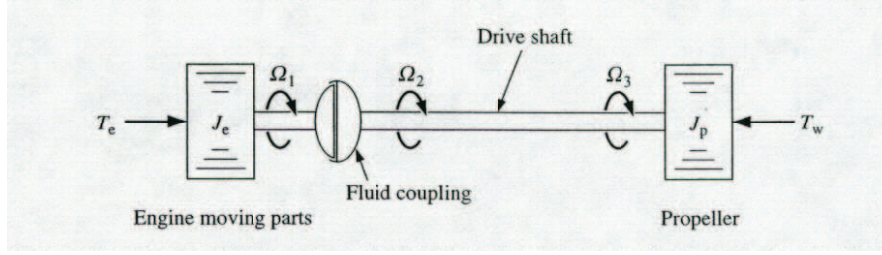


Figure 4: Schematic diagram of a simplified model of a ship propulsion system.

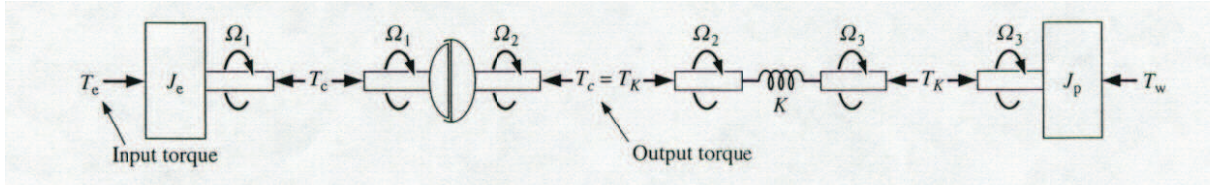


Figure 5: Diagram of a mass-spring-damper system in a gravity field.

The objective of this problem is to develop the equations for a math model for this system that would enable one to relate the shaft torque  $T_K(t)$  to the inputs  $T_e(t)$  and  $T_w(t)$ .

The free-body diagram for this system is shown in Figure 5.

For the engine (including moving parts and flywheel lumped together into an ideal inertia in which friction is ignored) we have

$$\frac{d\Omega_{1g}(t)}{dt} = \left(\frac{1}{J_e}\right)(T_e(t) - T_c(t)). \quad (17)$$

For the fluid coupling (with negligible inertia) we have

$$T_c(t) = C_c(\Omega_{1g}^2(t) - \Omega_{2g}^2(t)). \quad (18)$$

At the junction between the fluid coupling and drive shaft,

$$T_c(t) = T_K(t). \quad (19)$$

For the drive shaft (ideal spring with negligible friction and inertia)

$$\frac{dT_K(t)}{dt} = K(\Omega_{2g}(t) - \Omega_{3g}(t)). \quad (20)$$



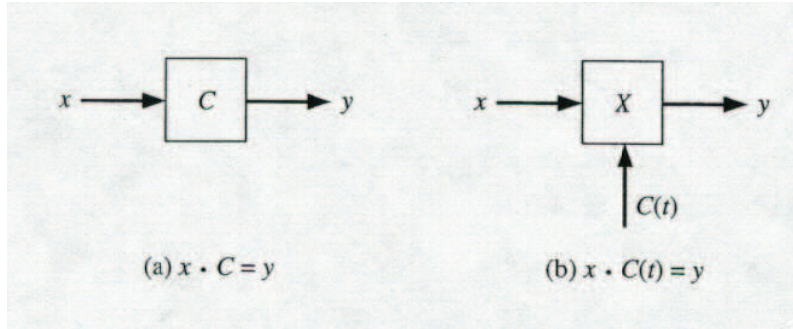


Figure 6: Coefficient block diagram for constant and time-varying coefficients.

For the propeller (ideal inertia with negligible friction) we have

$$\frac{d\Omega_{3g}(t)}{dt} = \left(\frac{1}{J_P}\right)(T_K(t) - T_w(t)). \quad (21)$$

Equations (17) through (21) constitute a necessary and sufficient set of five equations for this system containing five unknowns:  $\Omega_{1g}(t)$ ,  $\Omega_{2g}(t)$ ,  $T_c(t)$ ,  $T_K(t)$  and  $\Omega_{3g}(t)$ . The three variables  $\Omega_{1g}(t)$ ,  $T_K(t)$  and  $\Omega_{3g}(t)$  represent the system states, with their dynamics described by equations (17), (20) and (21). Equations (18) and (19) are static relations that can be used to eliminate the two additional variables  $\Omega_{2g}(t)$  and  $T_c(t)$  in terms of the state variables.

## Block Diagrams

One of the most useful forms of representing dynamic system models is through block diagrams. Figures 6 through 8 show some of the most widely used block diagram building blocks.

As an example let us develop the block diagram for the system shown in Figure 9.

Applying Newton's second law to the mass  $m$  yields

$$\frac{dv_{1g}(t)}{dt} = \frac{1}{m}(F_i(t) - kx_1(t) - bv_{1g}(t)). \quad (22)$$

We can now write the velocity and displacement of the mass  $m$  as

$$v_{1g}(t) = \int_{0^-}^t \left(\frac{dv_{1g}(t)}{dt}\right)dt + v_{1g}(0^-), \quad (23)$$



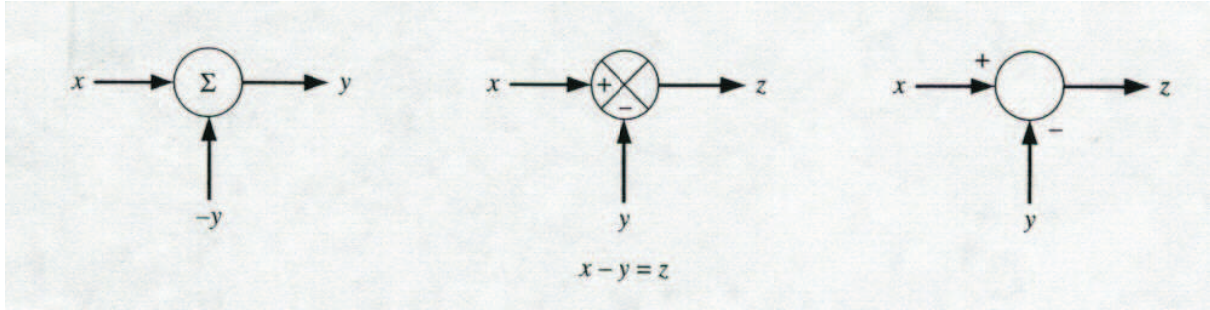


Figure 7: Commonly used summation block diagrams.

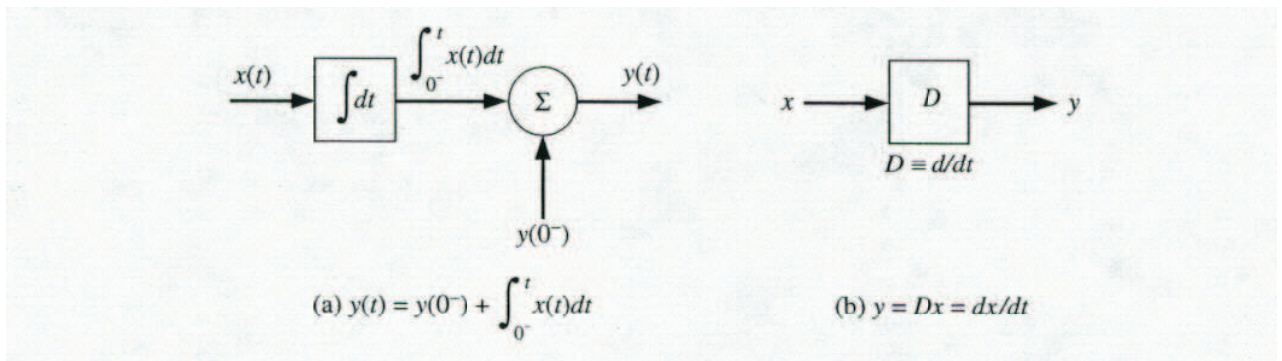


Figure 8: Block diagrams for integration and differentiation.

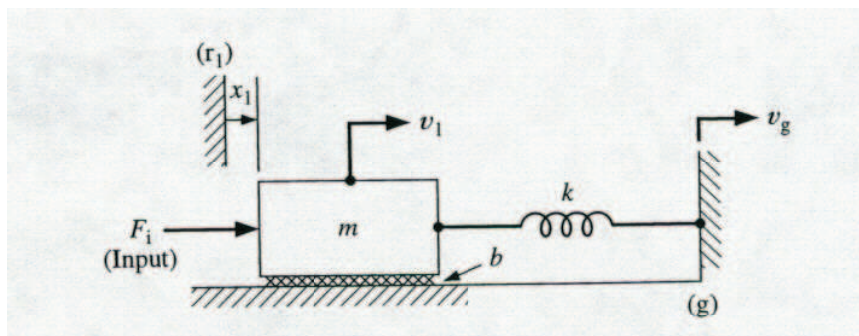


Figure 9: Simple mass-spring-damper system.

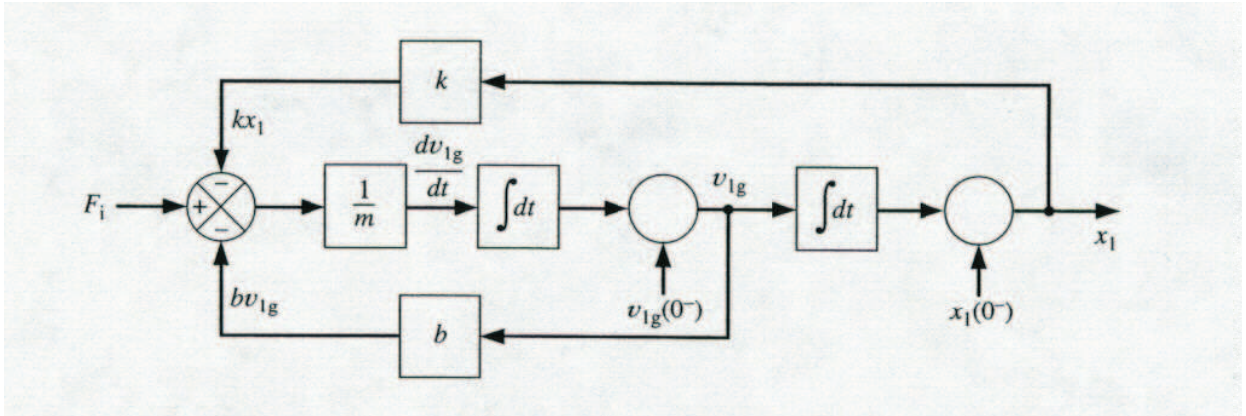


Figure 10: Block diagram of the simple mass-spring-damper system.

and

$$x_1(t) = \int_{0^-}^t (v_{1g}(t))dt + x_1(0^-). \quad (24)$$

The appropriate block diagram for this problem is shown in Figure 10.

### Reading Assignment

See separate file on textbook reading assignments depending on the text edition you own. Read the examples Handout E.6 posted on the course web page.